

# Diploma Thesis

## The Lagrangian Exploration Module

*Generation of homogeneous and isotropic turbulence with little mean flow for Lagrangian experiments*

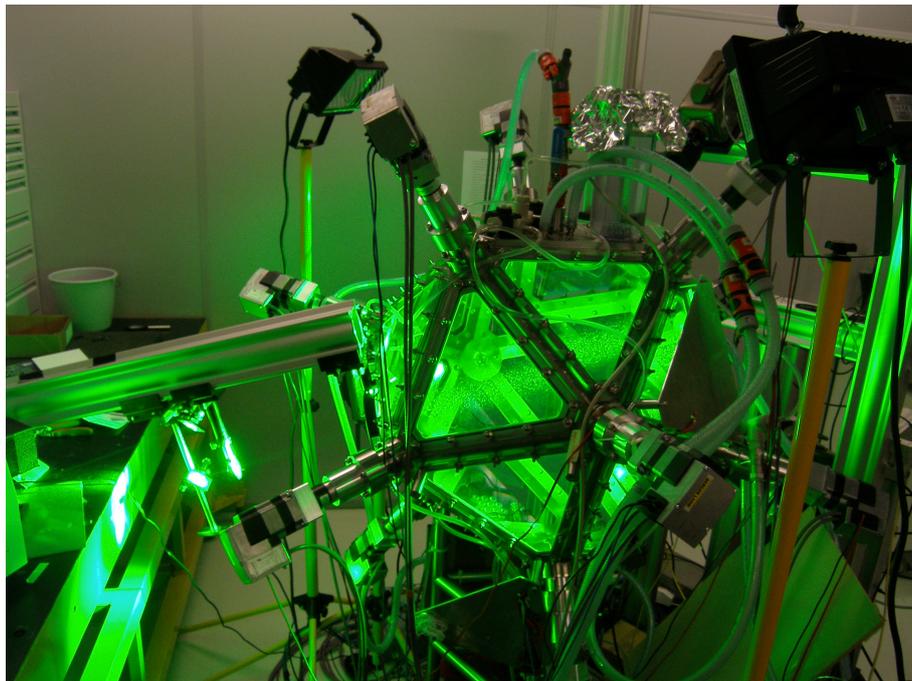
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# I Abstract

We designed and built an apparatus that generates nearly homogeneous and isotropic turbulence with a small mean flow. The apparatus is shaped as an icosahedron, and contains 140 liters of water. The flow is driven by 12 independently controlled propellers, each at one of the vertices of the icosahedron. We carried out Lagrangian particle tracking measurements experiments with measurement volumes up to a size of  $10 \times 10 \times 10$  [cm<sup>3</sup>], which is comparable with the integral length scale. The results show that the apparatus generates nearly homogeneous and isotropic turbulence with a Taylor microscale Reynolds number  $R_\lambda \approx 350$ . The measured mean flow is less than 20% of the fluctuating velocity throughout the observed volume. We also measured the Lagrangian structure functions of position and velocity in the flow and compared the measurements with theoretical predictions from Zybin et al [1].



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## II Introduction

Turbulence is a phenomenon which we encounter every day. Starting from the way vapor rises from a hot cup of coffee, to the mixing of milk and coffee caused by stirring, to the combustion process in a car engine, to pipe flows and a thunderstorm in the evening, turbulence plays a major role.

Predominant in every flow are inertia forces and damping frictional, or viscous, forces. If the frictional forces are stronger, for example in a slow moving or very viscous fluid, the flow is organized in parallel layers, which do not cross with each other. In this regime the flow is called laminar. An increase in the flow speed leads to an increase in the importance of inertia. When this happens, perturbations arising from the boundaries (e.g. propellers) or from body forces acting on the flow are not damped out and sum up until the layers break up into eddies of different sizes, which are advected by the mean flow. The flow is then called turbulent.

Although the equations that describe a flow are basically known since the time of Navier and Stokes, it is still not possible to predict the behavior of turbulent flows. The reason for this lies in the fact that these equations are hardly ever analytically solvable. In 1941, Kolmogorov [2] established a statistical approach to describe turbulence. In this approach the flow is characterized by means of distribution and autocorrelation functions. Due to his work and that of his successors it is possible to estimate the shape of these functions for a sufficiently turbulent flow. Although the original idea was formulated in the Eulerian framework, i.e. by observing a fixed region in the flow, new theoretical insights (see e.g. [3, 4]) were gained from models using the Lagrangian framework. The latter describes the flow by trajectories of fluid elements or particles.

One would like to test these theories but the difficulty lies in that they are only applicable where the turbulence is statistically homogenous and isotropic. The creation of experimental conditions fulfilling these two requirements is already a hard task. In addition, further problems arise from the measurement techniques available.

For a long time only Eulerian measurements were accessible to the experimentalist. Since 1971 [5], it is possible to follow fluid elements in a comoving coordinate system by introducing particles in the flow. These Lagrangian measurement techniques have enabled scientists to gain better understanding of processes such as the mixing of two different fluids or the properties of scalar fields like temperature [6] or pressure [7]. However, to follow particles for sufficiently long times, Lagrangian measurement techniques require the mean flow to be small compared to the fluctuations in the particles' velocity.

This requirement, however, could not be met by apparatuses such as wind tunnels or mixers. Additional problems for typical mixers in testing Kolmogorov's theories

lie in the absence of local isotropy and homogeneity over a sufficiently large observation volume.

In the current research project, the Lagrangian Exploration Module (LEM) will be presented as an apparatus meeting both isotropy and homogeneity requirements over a region comparable to the size of the biggest eddies. Furthermore, the LEM has a small mean-flow, thus providing ideal conditions for Lagrangian experiments on turbulent flows. To achieve these properties, the apparatus is shaped as an icosahedron containing 140 liters of water, and the flow is driven by twelve independently controlled propellers, each at one of the vertices of the icosahedron. This setup enables us to also investigate the influence of different forcings on the flow.

Lagrangian Particle Tracking measurements were performed on the LEM to characterize the machine. Parts of the results were compared with a prediction on the Lagrangian structure functions from Zybin *et al* [1]. Furthermore, we compared four different types of driving the flow. For two of the four forcings, a detailed study on their influence on the mean flow was carried out.

The present work is organized as follows:

Chapter III and IV provide the theoretical and experimental background. The experimental setup is explained in chapter V and the analysis methods are described in chapter VI. The results are subdivided into three main parts. Basic properties of the apparatus, such as Reynolds number and velocities are discussed in section VII.3. The influence of different forcings is investigated in section VII.4, and the predictions of Zybin *et al* [1] on the shape of the Lagrangian structure functions of position and velocity are compared with our experimental results in section VII.5.

A resume of the results is provided in chapter VIII.

## III Theoretical Background

### III.1 Describing a flow

Equations of motion for a single molecule and the interactions between several molecules are known since the time of Newton. Describing a simple flow from a molecular point of view is, however, not practical due to the large number of molecules present. One, therefore, turns to a continuum approach, as employed for example by Euler, Navier and Stokes.

All physical properties of the flow are then described by a field  $\mathbf{A}(\mathbf{X}, t)$  or  $\phi(\mathbf{X}, t)$ , where the fields are assumed to be smooth.

The following is mostly based on references [8] and [9]. Further details can also be found under [10], [11] and [12].

### III.2 Hydrodynamics

#### III.2.1 Continuum equation

In classical physics mass is conserved. For fluid dynamics mass conservation can be expressed in terms of the density  $\rho$ :

$$\frac{\partial}{\partial t} \int_V \rho dV = \int_{\partial V} \rho \mathbf{u} dA \quad (\text{III.1})$$

where  $\mathbf{u}$  is the flow through the surface  $\partial V$  of the volume  $V$ .

Using Gauss' Law we get

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho dV &= \int_V \nabla \cdot (\rho \mathbf{u}) dV \\ \Leftrightarrow \frac{\partial}{\partial t} \rho &= \nabla \cdot (\rho \mathbf{u}) \end{aligned} \quad (\text{III.2})$$

A constant density reduces equation (III.2) to

$$\nabla \cdot \mathbf{u} = 0 \quad (\text{III.3})$$

Consequently there are neither sinks nor sources in flows of incompressible fluids.

### III.2.2 Euler equation

In the middle of the 18th century Euler introduced the idea of internal pressure. Using this concept one can consider a small box with an edge length  $\Delta x$  and calculate the forces acting upon it. In the non-viscous case only pressure and inertial force apply. To conserve momentum both forces have to be equal.

$$\begin{aligned} \overbrace{\Delta p_i \cdot (\Delta x)^2}^{F_{\text{pressure}}} &= \overbrace{\rho (\Delta x)^3 \frac{d}{dt} u_i}^{F_{\text{inertial}}} \\ \Leftrightarrow \frac{\Delta p_i}{\Delta x} &= \rho \frac{d}{dt} u_i \end{aligned} \quad (\text{III.4})$$

Going to infinitesimally small sizes one can write

$$\rho \frac{d}{dt} \mathbf{u}(\mathbf{x}, t) = \nabla p \quad (\text{III.5})$$

The total derivative  $\frac{d}{dt} \phi(\mathbf{x}, t)$  of an arbitrary field depending only on position and time is

$$\begin{aligned} \frac{d}{dt} \phi(\mathbf{x}, t) &= \frac{\partial}{\partial t} \phi(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}} \phi(\mathbf{x}, t) \frac{\partial \mathbf{x}}{\partial t} \\ &= \left[ \frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \right] \phi(\mathbf{x}, t) \end{aligned} \quad (\text{III.6})$$

The term in brackets is called the comoving time derivative or the substantial time derivative. Applying equation (III.6) to equation (III.5) one gets

$$\rho \left( \frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \right) \mathbf{u}(\mathbf{x}, t) = \nabla p \quad (\text{III.7})$$

The result (III.7) is a nonlinear equation. Frictional forces, that have been neglected so far, will introduce further terms to (III.7). As friction becomes relevant when calculating drag forces, it is necessary to expand the considerations above to viscous fluids. Details of the derivation are presented in the following section.

### III.2.3 Navier–Stokes equation

The Euler equation (III.7) can be improved by refining the pressure force term. The surface forces acting on the small box, introduced in the section before, originate from the molecular forces inside the box. Body forces acting on the fluid can be expressed as a potential  $\Psi(\mathbf{x})$ . Using the stress tensor  $\tau_{ij}$  – known from solid state physics – we get<sup>1</sup>

$$\rho \frac{d u_j}{dt} = \frac{\partial \tau_{ij}}{\partial x_i} + \rho \frac{\partial \Psi}{\partial x_j} \quad (\text{III.8})$$

<sup>1</sup>Repeated indices are implicitly summed over (Einstein notation).

The stress tensor can be separated into a pressure,  $p$ , and a stress term,  $s_{ij}$ :

$$\tau_{ij} = p\delta_{ij} + s_{ij} \quad (\text{III.9})$$

We will focus on so called Newtonian fluids, e.g. water. They are characterized by the fact that the fluid is isotropic, with the stress,  $s_{ij}(\mathbf{r}, t)$ , vanishing for the fluid at rest and depending only on the strain rate at the same point  $\mathbf{r}$  and at the same time  $t$ . For small strain rates,  $\frac{\partial u_k}{\partial x_i}$ , we can assume that  $s_{ij}(\mathbf{r}, t)$  does only depend linearly on spatial derivatives,  $\frac{\partial u_k}{\partial x_i}$ .

$s_{ij}$  can be decomposed into a symmetric,  $\frac{1}{2}(s_{ij} + s_{ji})$ , and an antisymmetric part,  $\frac{1}{2}(s_{ij} - s_{ji})$ . If the fluid is rotating as a whole there is no relative motion between fluid elements, i.e. the stress vanishes. However, a fluid with a uniform rotation of angular velocity  $\omega$  has the velocity  $v(\mathbf{r}) = \omega \times \mathbf{r}$ . Applying this field to the symmetric and antisymmetric part one can easily see that only the symmetric part of  $s_{ij}$  vanishes. Consequently, it is

$$s_{ij} = s_{ji} \quad (\text{III.10})$$

The most general linear symmetric combination of  $\frac{\partial u_k}{\partial x_i}$  for an isotropic fluid is:

$$s_{ij} = a \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) + b \left( \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \quad (\text{III.11})$$

Equation (III.11) can be rewritten to

$$s_{ij} = \eta \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \zeta \left( \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \quad (\text{III.12})$$

Its derivative is

$$\begin{aligned} \frac{\partial}{\partial x_j} s_{ij} &= \eta \left( \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \zeta \left( \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \\ &= \eta \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \eta \left( \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} - \frac{2}{3} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \zeta \left( \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \\ &= \eta \frac{\partial^2 u_i}{\partial x_j^2} + \eta \left( \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} - \frac{2}{3} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) + \zeta \left( \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) \\ &= \eta \frac{\partial^2 u_i}{\partial x_j^2} + \left( \frac{\eta}{3} + \zeta \right) \left( \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) \end{aligned} \quad (\text{III.13})$$

The parameters  $\eta$  and  $\zeta$  are called viscosities, as an approximation they are assumed to be independent of velocity, position and time. Inserting the obtained stress tensor in equation (III.8) we get

$$\begin{aligned} \rho \frac{d u_j}{dt} &= \frac{\partial p}{\partial x_i} \delta_{ij} + \eta \frac{\partial^2 u_j}{\partial x_j^2} + \left( \frac{\eta}{3} + \zeta \right) \left( \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_k} \right) + \rho \frac{\partial}{\partial x_j} \Psi \\ \Leftrightarrow \rho \frac{d}{dt} \mathbf{u} &= \nabla p + \eta \nabla^2 \mathbf{u} + \left( \frac{\eta}{3} + \zeta \right) \nabla (\nabla \cdot \mathbf{u}) + \rho \mathbf{f} \end{aligned} \quad (\text{III.14})$$

This is the Navier–Stokes equation. It should be noted that  $\eta$  and  $\zeta$  are depending on the temperature. Thus, further information is needed to fully describe a flow. For constant density fluids the divergence of  $\mathbf{u}$  vanishes, so that the Navier–Stokes equation turns into

$$\rho \frac{d}{dt} \mathbf{u} = \nabla p + \eta \nabla^2 \mathbf{u} + \mathbf{f} \quad (\text{III.15})$$

### III.2.4 The nondimensional Navier-Stokes equation

Usually the Navier-Stokes equation is presented in a dimensionless form. The advantage of such a representation is that different hydrodynamical setups can be compared. So long as their relative dimensions correspond, they'll be described by the same (dimensionless) equation. Accordingly their flow behavior should be the same.

Introducing a length scale,  $\mathcal{L}$ , and a velocity scale,  $\mathcal{U}$ , we can write

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\mathcal{L}} \quad \text{and} \quad \hat{t} = t \frac{\mathcal{L}}{\mathcal{U}} \quad (\text{III.16})$$

as well as

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{\rho \mathcal{U}^2} \quad \text{and} \quad \hat{\mathbf{u}} = \frac{\mathbf{u}}{\mathcal{U}} \quad (\text{III.17})$$

Plugging these new parameters into equation (III.15) one obtains:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \quad (\text{III.18})$$

where  $Re$  is the so called Reynolds number

$$Re = \frac{\rho \mathcal{U} \mathcal{L}}{\eta} = \frac{\mathcal{U} \mathcal{L}}{\nu} \quad (\text{III.19})$$

In turbulence research the kinematic viscosity  $\nu = \frac{\eta}{\rho}$  is often used.

As already indicated, setups having the same boundary conditions and the same Reynolds number develop the same flow characteristics. The Reynolds number can be seen as the ratio between the magnitude of inertia forces arising from  $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$  and the viscous forces represented by  $\nabla^2 \mathbf{u}$ . For sufficiently high Reynolds number the flow is dominated by inertia and frictional forces are negligible. Moreover, one obtains the Euler equation (III.7) in the limit of  $Re \rightarrow \infty$ .

## III.3 Introduction to turbulent flows

Due to its non-linear term the Navier-Stokes equation (III.15) is sensitive to initial conditions and boundary conditions. However, for a small Reynolds number  $Re$  the viscous term of the Navier-Stokes equation dominates. Small perturbations arising from the boundaries or body forces acting on the fluid are damped out, thus the fluid flows in parallel layers with no disruption between the layers. This type of flow is called laminar.

For an increasing Reynolds number these perturbations are not damped out anymore and the layers to break up into many eddies, which are advected with the mean-flow. The system evolves into a chaotic system which is called turbulence. In this regime mixing is enhanced, but also the drag force acting a body moving through the flow is higher in a turbulent flow than in the laminar case.

For an easier characterization, it is useful to compose the velocity field into a mean flow field  $\langle \mathbf{u}(\mathbf{x}) \rangle$  plus the turbulent fluctuations of the velocity  $\mathbf{u}'(\mathbf{x}, t)$ :

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}) \rangle + \mathbf{u}'(\mathbf{x}, t) \quad (\text{III.20})$$

The theories referred to in this text assume a negligible mean flow.

## III.4 Energy cascade

The description of turbulence relies very much on one concept first introduced by Richardson: the energy cascade.

Starting a flow through e.g. propellers creates big whirls of a size,  $L$ , and a Reynolds number,  $Re(L)$ . For sufficiently strong forcing  $Re(L)$  at this scale  $L$  is big enough to neglect viscosity and resulting energy losses. However, the injected energy has to be dissipated.

According to Richardson each big vortex then breaks up into several smaller vortices. The latter would be of a different length scale  $l$  and more importantly of a smaller Reynolds number  $Re(l) < Re(L)$ . The child vortices again split into more vortices of even smaller size  $l'$  and  $Re(l') < Re(l)$ . This self-similar process is repeated until molecular interactions start playing a role.

In 1941, Kolmogorov refined this model by making three hypotheses. According to these, energy conservation dictates that the rate of energy transferred from a length scale to its child vortices is the same at all scales in the energy cascade. Thus, one can define the energy dissipation rate,  $\varepsilon$ , as the dissipated energy per unit time and unit mass. Since the transferred energy equals the dissipated energy,  $\varepsilon$ , is also called energy transfer rate.

Furthermore, he postulated that the size of the smallest eddies depends only on the viscosity of the fluid and the energy transfer rate. It is common to use  $\eta$  to indicate the smallest (Kolmogorov) scale; to avoid confusion the kinematic viscosity,  $\nu = \frac{\eta}{\rho}$ , instead of the dynamical viscosity,  $\eta$ , will be used from now on.

Kolmogorov also stated, that the statistical behavior of eddies much smaller than the biggest length scales,  $L$ , but still much bigger than the smallest eddies depends only on the energy transfer rate,  $\varepsilon$ . This range is called inertial range. A size characteristic for this range is the Taylor micro-scale,  $\lambda$ . The application of his hypotheses will be shown in section III.6.

A visualization of the Richardson cascade is provided in figure III.1.

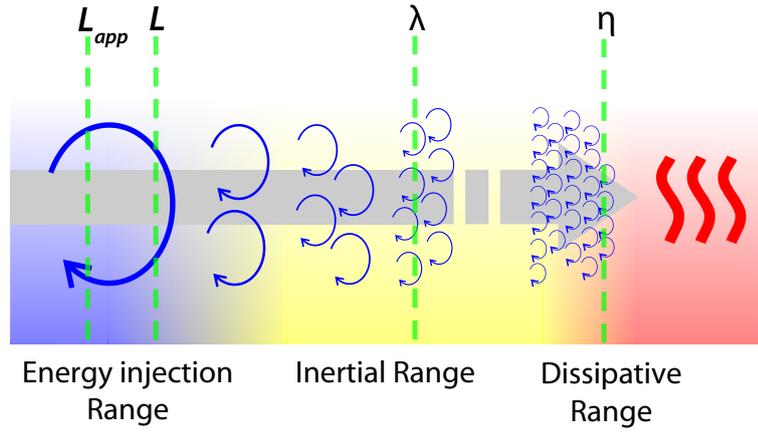


Figure III.1: Sketch of the Richardson cascade. The system (of size  $L_{app}$ ) injects energy at the size  $L$  characterizing the biggest whirls. These whirls break up into smaller whirls, which break up into even smaller eddies, and so forth. This process is stopping in the dissipative range when viscosity becomes non-neglectible and the injected energy is finally converted to heat.

## III.5 Dimensional arguments

Dimensional arguments provide a useful tool for order of magnitude estimates. The Navier-Stokes equation (III.15) is fully described by length, velocity, pressure, density and viscosity.

For example, the characteristic time scale,  $T$ , for the big whirls with size  $\mathcal{L}$  and characteristic velocity,  $\mathcal{U}$ , is

$$T \sim \frac{\mathcal{L}}{\mathcal{U}} \quad (\text{III.21})$$

Further, the energy dissipation rate  $\varepsilon$  is measured in  $\left[ \frac{kg\ m^2}{s^2} \cdot \frac{1}{kg} \cdot \frac{1}{s} \right] = \left[ \frac{m^2}{s^3} \right]$  and is fully characterized by the velocity and the length scale of turbulent eddies. The only possible combination using only these parameters with the same units is:

$$\varepsilon \sim \frac{\mathcal{U}^3}{\mathcal{L}} \sim \frac{\langle \mathbf{u}^2 \rangle^{3/2}}{\mathcal{L}} \quad (\text{III.22})$$

## III.6 K41

In 1941, Kolmogorov [2] derived important properties of turbulent flows using Richardson's idea of the energy cascade. His publications had such a big impact on the turbulence community that his theory is referred to as K41.

### III.6.1 Kolmogorov's hypotheses & scales

Before Kolmogorov's hypotheses can be introduced one has to define local homogeneity and isotropy.

Looking at region  $\mathcal{G}$  in a turbulent flow, let  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}$  represent a set of points within  $\mathcal{G}$ . Choosing one of the  $\mathbf{x}^{(i)}$  as a reference  $\mathbf{x}^{(0)}$ , one can introduce new coordinates

$$\mathbf{y}^{(i)} = \mathbf{x}^{(i)} - \mathbf{x}^{(0)}$$

and velocity differences

$$\mathbf{v}(\mathbf{y}^{(i)}) = \mathbf{U}(\mathbf{x}^{(i)}, t) - \mathbf{U}(\mathbf{x}^{(0)}, t)$$

Every  $\mathbf{v}(\mathbf{y}^{(i)})$  is itself a random variable following a probability density function dependent on the flow properties at  $\mathbf{y}^{(i)}$ . Allowing  $f_v \equiv f_v(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(N)})$  to designate the joint probability density function (PDF) of finding a given set of velocities  $\mathbf{v}(\mathbf{y}^{(1)}), \dots, \mathbf{v}(\mathbf{y}^{(N)})$  at the  $N$  points and the same time, the following definitions are possible:

**local homogeneity** A turbulent flow is locally homogeneous in  $\mathcal{G}$ , if, for a set  $\mathbf{y}^{(n)} (n = 1, \dots, N)$ , the PDF  $f_v$  is independent of the exact choice of the reference point  $\mathbf{x}^{(0)}$ .

**local isotropy** The turbulent flow is locally isotropic in  $\mathcal{G}$  if it is locally homogeneous and invariant to rotations and reflections.

With the help of these definitions Kolmogorov established three hypotheses

**hypothesis of local isotropy** In any turbulent flow with a sufficiently high Reynolds number, the turbulence is – to a good approximation – locally isotropic for a sufficiently small  $\mathcal{G}$  not situated near the boundaries or singularities of the flow.

**first similarity hypothesis** For locally isotropic turbulent flows  $f_v$  is only determined by the viscosity,  $\nu$ , and the energy dissipation rate,  $\varepsilon$ .

**second similarity hypothesis** If the separation between the vectors  $\mathbf{y}^{(m)}$  is large compared to the Kolmogorov scale,  $\eta$ , the joint PDF  $f_v$  is uniquely determined by the dissipation rate,  $\varepsilon$ , and does not depend on  $\nu$ .

Kolmogorov realized that, according the Richardson cascade model, the smallest scales of motion occur at a Reynolds number equal to 1.

Using the first similarity hypothesis and equation (III.22) one can define the smallest length scale  $\eta$ , the smallest velocity scale  $u_\eta$  and the smallest time scale  $\tau_\eta$ :

$$\begin{aligned} 1 = Re &= \frac{\eta u_\eta}{\nu} = \frac{\eta^{4/3} \varepsilon^{1/3}}{\nu} \\ \Rightarrow \eta &= (\nu^3 / \varepsilon)^{1/4} \end{aligned} \quad (\text{III.23})$$

Similarly for the smallest velocity scale,  $u_\eta$ , one arrives at:

$$\begin{aligned} 1 = Re &= \frac{\eta u_\eta}{\nu} = \frac{u_\eta^4}{\varepsilon \nu} \\ \Rightarrow u_\eta &= (\varepsilon \nu)^{1/4} \end{aligned} \quad (\text{III.24})$$

And for the smallest time scale,  $\tau_\eta$ , one deduces:

$$\tau_\eta = \frac{\eta}{u_\eta} = (\nu/\varepsilon)^{1/2} \quad (\text{III.25})$$

Or written in terms of the Reynolds number

$$\eta/l_0 \propto Re^{-3/4} \quad (\text{III.26})$$

$$u_\eta/u_0 \propto Re^{-1/4} \quad (\text{III.27})$$

$$\tau_\eta/\tau_0 \propto Re^{-1/2} \quad (\text{III.28})$$

One generally speaks of  $\eta$ ,  $u_\eta$  and  $\tau_\eta$  as Kolmogorov scales.

### III.6.2 Second order structure functions – $D_{LL}$ and $D_{NN}$

The Eulerian second order structure function is defined as

$$D_{ij}(\mathbf{r}, \mathbf{x}, t) = \langle [U_i(\mathbf{x} + \mathbf{r}, t) - U_i(\mathbf{x}, t)] [U_j(\mathbf{x} + \mathbf{r}, t) - U_j(\mathbf{x}, t)] \rangle \quad (\text{III.29})$$

where the index refers to one component of the velocity vector  $\mathbf{U}$  and the brackets  $\langle \cdot \rangle$  denote the average over several similarly prepared experiments.

In terms of  $\mathbf{y}$  and  $\mathbf{v}$  i.e. substituting  $\mathbf{r} = \mathbf{y}^{(2)} - \mathbf{y}^{(1)}$ ,  $\mathbf{x} = \mathbf{y}^{(1)} + \mathbf{x}^{(0)}$  and choosing the origin to be  $\mathbf{x}^{(0)}$ :

$$D_{ij}(\mathbf{y}^{(2)} - \mathbf{y}^{(1)}, \mathbf{y}^{(1)} + \mathbf{x}^{(0)}, t) = \langle [v_i(\mathbf{y}^{(2)}) - v_i(\mathbf{y}^{(1)})] [v_j(\mathbf{y}^{(2)}) - v_j(\mathbf{y}^{(1)})] \rangle \quad (\text{III.30})$$

Assuming local isotropy – i.e.  $r \equiv \|\mathbf{r}\|$  much smaller than the biggest length scale – then  $D_{ij}$  is neither dependent on its second argument  $\mathbf{x}$  in equation (III.29) nor on  $\mathbf{x}^{(0)}$  in equation (III.30). It only depends on  $\mathbf{r}$  and can be written  $D_{ij}(\mathbf{r}, t)$ . Seeing that it only depends on  $\mathbf{r}$ ,  $D_{ij}$  can be split up into a term containing the inter-dependence between the vector components  $r_i$  and  $r_j$  and a term independent of these.

Such a representation would be

$$D_{ij} = A(r, t)\delta_{ij} + B(r, t)\frac{r_i r_j}{r r}$$

with two (yet) arbitrary functions  $A(r, t)$  and  $B(r, t)$ . It is convenient to rotate the coordinate system such that  $\mathbf{r}$  is  $r \cdot \mathbf{e}_x$ . One obtains:

$$D_{LL} \equiv D_{xx} = A(r, t) + B(r, t) = \langle (\Delta v_x)^2 \rangle \quad (\text{III.31})$$

$$D_{ij} = B(r, t)\frac{r_i r_j}{r r} = 0 \quad \text{with } i \neq j \quad (\text{III.32})$$

$$D_{NN} \equiv D_{yy} = D_{zz} = A(r, t) = \langle (\Delta v_y)^2 \rangle = \langle (\Delta v_z)^2 \rangle \quad (\text{III.33})$$

Combining the equations above we arrive at

$$D_{ij}(\mathbf{r}, t) = D_{NN}(r, t)\delta_{ij} + [D_{LL}(r, t) - D_{NN}(r, t)]\frac{r_i r_j}{r^2} \quad (\text{III.34})$$

In order to relate  $D_{NN}$  and  $D_{LL}$  consider again equation (III.30). Rewriting  $v_i^{(k)} \equiv v_i(\mathbf{y}^k)$  for  $k = 1$  or  $2$  and expanding equation (III.30) one calculates:

$$D_{ij}(\mathbf{r}, t) = \langle v_i^{(1)} v_j^{(1)} \rangle + \langle v_i^{(2)} v_j^{(2)} \rangle - \langle v_i^{(1)} v_j^{(2)} \rangle - \langle v_i^{(2)} v_j^{(1)} \rangle \quad (\text{III.35})$$

Because of isotropy  $\langle v_i^{(1)} v_j^{(1)} \rangle = \langle v_i^{(2)} v_j^{(2)} \rangle$ . By mirroring the coordinate system we get  $\langle v_i^{(1)} v_j^{(2)} \rangle = \langle v_i^{(2)} v_j^{(1)} \rangle$ . Thus

$$D_{ij}(\mathbf{r}, t) = 2 \langle v_i^{(1)} v_j^{(1)} \rangle - 2 \langle v_i^{(1)} v_j^{(2)} \rangle = \frac{2}{3} \langle \mathbf{v}^2 \rangle - 2 \langle v_i^{(1)} v_j^{(2)} \rangle \quad (\text{III.36})$$

The derivative of (III.36) yields:

$$\frac{\partial D_{ij}}{\partial y_j^{(2)}} = -2 \left\langle v_i^{(1)} \frac{\partial v_j^{(2)}}{\partial y_j^{(2)}} \right\rangle$$

The continuity equation dictates:

$$\frac{\partial v_j^{(2)}}{\partial y_j^{(2)}} = 0 \quad \Rightarrow \quad \frac{\partial D_{ij}}{\partial y_j^{(2)}} = 0$$

The same holds true for the derivative  $\frac{\partial D_{ij}}{\partial y_j^{(1)}}$ . Hence<sup>2</sup>:

$$\begin{aligned} 0 &= \frac{\partial D_{ij}(\mathbf{r}, t)}{\partial r_i} = \frac{\partial}{\partial r_i} (D_{NN}(r, t) \delta_{ij}) + \frac{\partial}{\partial r_i} \left( [D_{LL}(r, t) - D_{NN}(r, t)] \frac{r_i r_j}{r^2} \right) \\ &= \frac{r_j}{r^2} \left[ r \frac{\partial D_{LL}(r, t)}{\partial r} + 2 (D_{LL}(r, t) - D_{NN}(r, t)) \right] \end{aligned} \quad (\text{III.37})$$

Therefore:

$$D_{NN} = D_{LL} + \frac{r}{2} \frac{\partial D_{LL}(r, t)}{\partial r} \quad (\text{III.38})$$

<sup>2</sup> Auxiliary calculations using Einstein summation convention:

$$\frac{\partial}{\partial r_i} (D_{NN}(r, t) \delta_{ij}) = \frac{\partial D_{NN}(r, t)}{\partial r_i} \delta_{ij} + D_{NN}(r, t) \frac{\partial \delta_{ij}}{\partial r_i} = \frac{\partial D_{NN}(r, t)}{\partial r} \frac{\partial r}{\partial r_j} = \frac{r_j}{r} \frac{\partial D_{NN}(r, t)}{\partial r}$$

$$\begin{aligned} \frac{\partial}{\partial r_i} (r_i r_j r^{-2}) &= \frac{\partial r_i}{\partial r_i} r_j r^{-2} + \frac{\partial r_j}{\partial r_i} r_i r^{-2} + \frac{\partial r^{-2}}{\partial r_i} r_i r_j \\ &= 3 r_j r^{-2} + r_i \delta_{ij} r^{-2} + \frac{\partial r^{-2}}{\partial r_i} r_i r_j \\ &= 3 r_j r^{-2} + r_j r^{-2} - 2 r_j r^{-2} \\ &= 2 \frac{r_j}{r^2} \end{aligned}$$

$$\frac{r_i r_j}{r^2} \frac{\partial}{\partial r_i} D_{LL}(r, t) - \frac{r_i r_j}{r^2} \frac{\partial}{\partial r_i} D_{NN}(r, t) = \frac{r_j}{r^2} \left( r \frac{\partial D_{LL}(r, t)}{\partial r} - r \frac{\partial D_{NN}(r, t)}{\partial r} \right)$$

For  $\mathbf{r}$  parallel to  $\mathbf{e}_x$  we have  $r_i r_j = 0$  for  $i \neq j$ . Hence, the two derivative terms of  $D_{NN}$  are the same and cancel each other.

In consequence,  $D_{NN}$  is completely determined by  $D_{LL}$ . Assuming a statistically stationary turbulent flow the brackets  $\langle \cdot \rangle$  denoting the average over identically or rather similarly prepared experiments can be replaced by a temporal average.

The term  $(\varepsilon r)^{2/3}$  has the unit of velocity squared and can be used to create a non-dimensional second-order structure function  $\hat{D}_{LL}(r/\eta)$ .

$$D_{LL}(r) = (\varepsilon r)^{2/3} \hat{D}_{LL}(r/\eta)$$

For large  $r/\eta$  the second similarity hypothesis claims that  $D_{LL}$  is independent of  $\nu$ . Since  $\eta$  is a function of  $\varepsilon$  and  $\nu$ , to satisfy the hypothesis  $\hat{D}$  cannot depend on  $r/\eta$ . I.e.  $\hat{D}_{LL} = \text{const} = C_2$ . Experiments have shown that  $C_2$  is approximately 2.1. Plugging that into equation (III.38) we get

$$D_{NN} = D_{LL} + \frac{1}{3}D_{LL} = \frac{4}{3}D_{LL} = \frac{4}{3}C_2(\varepsilon r)^{2/3} \quad (\text{III.39})$$

Equation (III.39) allows to measure the energy dissipation rate by investigating the compensated second order structure function

$$\varepsilon(r) \approx \left( \frac{D_{LL}}{C_2} \right)^{3/2} \frac{1}{r} = \left( \frac{3 D_{NN}}{4 C_2} \right)^{3/2} \frac{1}{r} \quad (\text{III.40})$$

For sufficiently turbulent flows equation (III.40) shows a plateau in the inertial range.

### III.6.3 Eulerian structure functions of order three and higher – $D_{LLL}$ and $D_p$

The Eulerian structure function of order  $p$  is defined as

$$D_p = \langle |v_L(\mathbf{x} + \mathbf{r}, t) - v_L(\mathbf{x}, t)|^p \rangle \quad (\text{III.41})$$

The second similarity hypothesis and dimensional analysis yields that  $D$  scales as

$$D_p = C_p (\varepsilon r)^{p/3} \quad (\text{III.42})$$

where  $C_p$  is a constant. It should be noted that the predicted scaling exponent of  $p/3$  for  $p \neq 3$  has not been observed experimentally.

For the third order, an exact solution can be derived from the Navier-Stokes equation:

$$D_{LLL}(r, t) = -\frac{4}{5}\varepsilon r \quad (\text{III.43})$$

Equation III.43 is called Kolmogorov  $\frac{4}{5}$  law.

A detailed discussion of the Kolmogorov  $\frac{4}{5}$  law as well as of higher order structure functions can be found in Frisch's book [10].

### III.6.4 Eulerian Autocorrelation

Let

$$R_{ij}(\mathbf{r}, t) \equiv \langle u_i(\mathbf{x} + \mathbf{r}, t) u_j(\mathbf{x}, t) \rangle \quad (\text{III.44})$$

denote the Eulerian autocorrelation. In the case of homogeneous isotropic turbulence with zero mean velocity  $R_{ij}$  is independent of  $\mathbf{x}$ . Using the same arguments and the same notation already used to obtain (III.34) we get

$$\begin{aligned} R_{ij}(\mathbf{r}, t) &= R_{NN}(r, t) \delta_{ij} + [R_{LL}(r, t) - R_{NN}(r, t)] \frac{r_i r_j}{r^2} \\ &= \langle u_L^2 \rangle \left( g(r, t) \delta_{ij} + [f(r, t) - g(r, t)] \frac{r_i r_j}{r^2} \right) \quad \text{with } \langle u_L^2 \rangle \equiv \langle (\mathbf{u} \cdot \mathbf{e}_r)^2 \rangle \end{aligned} \quad (\text{III.45})$$

where  $f \equiv R_{LL}/\langle u_L^2 \rangle$  is the longitudinal and  $g \equiv R_{NN}/\langle u_L^2 \rangle$  is the transverse autocorrelation functions. In conformity with the continuity equation the following relation has to be valid:

$$\frac{\partial R_{ij}}{\partial r_j} = \frac{\partial}{\partial r_j} \langle u_i(\mathbf{x} + \mathbf{r}, t) u_j(\mathbf{x}, t) \rangle = \langle u_i(\mathbf{x} + \mathbf{r}, t) \frac{\partial}{\partial r_j} u_j(\mathbf{x}, t) \rangle = 0 \quad (\text{III.46})$$

Analogous to equation (III.38) one deduces:

$$g(r, t) = f + \frac{r}{2} \frac{\partial f(r, t)}{\partial r} \quad (\text{III.47})$$

Two distinct length scales can be defined from the autocorrelation  $f$ : the integral length scale,  $L$ , and the Taylor micro scale,  $\lambda$ . Since transverse and longitudinal functions are connected by equation (III.47) these scales can also be computed from  $g$ .

### III.6.5 Integral Length Scale

The longitudinal integral length scale is defined as

$$L_{LL} \equiv \int_0^\infty f(r, t) dr \quad (\text{III.48})$$

whereas the transverse integral length scale is given by

$$\begin{aligned} L_{NN} &\equiv \int_0^\infty g(r, t) dr \\ &= \int_0^\infty f(r, t) dr + \int_0^\infty \frac{1}{2} r \frac{\partial}{\partial r} f(r, t) dr \\ &= L_{LL} + \left[ \frac{1}{2} r f(r, t) \right]_0^\infty - \int_0^\infty \frac{1}{2} f(r, t) dr \end{aligned} \quad (\text{III.49})$$

If  $f(r, t)$  decays more rapidly than  $r^{-2}$ , the term  $\left[\frac{1}{2}rf(r, t)\right]_0^\infty$  vanishes and hence

$$L_{NN} = \frac{1}{2}L_{LL} \quad (\text{III.50})$$

The longitudinal integral length scale  $L_{LL}$  can be considered to be the scale of the biggest eddies. Since longitudinal and transverse integral length scale are related, it is common to use the integral length scale  $L = L_{LL}$ .

### III.6.6 Taylor microscale & Taylor scale Reynolds number – $R_\lambda$

The second length scale which can be obtained from the autocorrelation  $f$  is the Taylor microscale  $\lambda_f(t)$ . Because of isotropy  $f(r)$  is an even function assuming values below or equal to unity. Thus the first derivative  $f'(0, t)$  at the origin has to be zero. Furthermore autocorrelation functions decrease at the origin. Consequently the origin will be a maximum and the second derivative  $f''(0, t)$  at that point will be non-positive. The Taylor microscale can then be defined as

$$\lambda_f(t) \equiv \left[-\frac{1}{2}f''(0, t)\right]^{-1/2} \quad (\text{III.51})$$

Using equation (III.47) it is

$$\begin{aligned} -\langle u^2 \rangle f''(0, t) &= -\langle u^2 \rangle \lim_{r \rightarrow 0} \frac{\partial^2}{\partial r^2} f(r, t) \\ &= -\lim_{r \rightarrow 0} \left\langle \left( \frac{\partial^2 u_L}{\partial r^2} \right)_{\mathbf{x}+\mathbf{e}_r} u_L(\mathbf{x}, t) \right\rangle \\ &= -\left\langle \left( \frac{\partial^2 u_L}{\partial x_L^2} \right) u_L \right\rangle \\ &= -\left\langle \frac{\partial}{\partial x_L} \left( u_L \frac{\partial u_L}{\partial x_L} \right) \right\rangle + \left\langle \left( \frac{\partial u_L}{\partial x_L} \right)^2 \right\rangle \\ &= -\frac{\partial}{\partial x_L} \left\langle \left( u_L \frac{\partial u_L}{\partial x_L} \right) \right\rangle + \left\langle \left( \frac{\partial u_L}{\partial x_L} \right)^2 \right\rangle \\ &= \left\langle \left( \frac{\partial u_L}{\partial x_L} \right)^2 \right\rangle \end{aligned} \quad (\text{III.52})$$

Combining (III.52) and (III.51) it is

$$\left\langle \left( \frac{\partial u_L}{\partial x_L} \right)^2 \right\rangle = \frac{2\langle u^2 \rangle}{\lambda_f^2(t)} \quad (\text{III.53})$$

$\lambda_g(t) \equiv \left[-\frac{1}{2}g''(0, t)\right]^{-1/2}$  is the transverse Taylor microscale, using equation (III.47) one can calculate the relation between longitudinal and transverse microscale.

$$\begin{aligned}
\lambda_g(t) &= \left[-\frac{1}{2}g''(0, t)\right]^{-1/2} \\
&= \left[-\frac{1}{2}\left(f'' + \lim_{r \rightarrow 0} \frac{r}{2} \frac{\partial^3 f(r, t)}{\partial r^3}\right)\right]^{-1/2} \\
&= \left[-\frac{1}{2}\left(f'' + \frac{1}{2} \lim_{r \rightarrow 0} \left[\frac{\partial}{\partial r} [r f''] - f''\right]\right)\right]^{-1/2} \quad (\text{III.54}) \\
&= \left[-\frac{1}{2}\left(f'' - \frac{1}{2}f''\right)\right]^{-1/2} \\
&= \frac{\lambda_f(t)}{\sqrt{2}}
\end{aligned}$$

For historic reasons, one usually talks about the Taylor microscale  $\lambda = \lambda_g$ .

It can be shown that

$$\varepsilon = 15 \nu \left\langle \left(\frac{\partial u_L}{\partial x_L}\right)^2 \right\rangle \quad (\text{III.55})$$

Using equation (III.54) and (III.53) it follows

$$\varepsilon = 15 \nu \frac{u'^2}{\lambda_g^2} \quad (\text{III.56})$$

This can be used to define the Taylor scale Reynolds number

$$R_\lambda \equiv \frac{u' \lambda_g}{\nu} \quad (\text{III.57})$$

The relation between  $Re$  and  $R_\lambda$  can be easily shown

$$\begin{aligned}
R_\lambda^2 &= \frac{u'^2 \lambda_g^2}{\nu^2} \\
&= 15 \nu \frac{u'^2}{\varepsilon} \cdot \frac{u'^2}{\nu^2} \\
&= 15 \frac{l}{u'^3} \frac{u'^4}{\nu} \\
&= 15 Re
\end{aligned} \quad (\text{III.58})$$

Thus,  $R_\lambda$  is the Reynolds number at the intermediate length scale  $\lambda_g$  lying between then  $\eta$  and  $L$ . There is no direct physical meaning of this length scale but that  $\lambda$  is well situated in the inertial range. In turbulence research it is common to use  $R_\lambda$  instead of  $Re$  as a measure for the strength of the turbulence. The magnitude of  $\lambda_g$  can be estimated by applying the definition of the Kolmogorov scales to equation (III.56). It is

$$\lambda_g = \sqrt{15} \tau_\eta u' \sim \sqrt{15} \eta R_\lambda^{1/2} \quad (\text{III.59})$$

For example for intense turbulence of  $R_\lambda \approx 660$  the Taylor micro scale is 100 times the Kolmogorov length scale but still only 1/44 of the integral length scale.

### III.7 Limitations

The results obtained in the section III.6 are based on Kolmogorov's hypotheses and the Richardson cascade. Only Kolmogorov's four-fifth law (equation III.43) can be derived directly from the Navier-Stokes equation. Therefore, some limitations arise.

Landau pointed out (see [9], page 176), that the averaging process makes it impossible to find universal laws for the structure functions. The ensemble average over  $n$  realizations of the turbulent flow of a structure function  $D_p$  is

$$D_p^{avg} = \frac{1}{N} \sum_{i=1}^N D_i \quad (\text{III.60})$$

and the average of their corresponding energy transfer rate  $\varepsilon_i$

$$\varepsilon^{avg} = \frac{1}{N} \sum_{i=1}^N \varepsilon_i \quad (\text{III.61})$$

Combining equations (III.42), (III.61) and (III.60) yields

$$\left( \frac{1}{N} \sum_{i=1}^N \varepsilon \right)^{p/3} = \frac{1}{N} \sum_{i=1}^N (\varepsilon_i)^{p/3} \quad (\text{III.62})$$

Equation (III.62) can only be true for the third order structure function. Therefore, the structure functions depend on the way the turbulence is created.

A further problem emerges from the Richardson cascade. In the model energy is strictly transferred from bigger whirled to their smaller child whirled. In nature smaller whirled can act also on bigger whirled. Furthermore, the child eddies do not necessarily occupy the same amount of volume than their parent eddy. This leads to a violation of the self similarity of the model.

Kolmogorov's requirements on the choice of  $\mathcal{G}$  and the Reynolds number are experimentally vague, but there is experimental evidence that the turbulence approaches local isotropy with increasing  $Re$ . Yet, at intense turbulence of  $R_\lambda \approx 970$  the flow is still locally anisotropic [13].

These limitations are usually referred to as internal intermittency. Newer models tried to improve K41 predictions, but we still lack of a good understanding of turbulence.

A locally isotropic flow regime is major hypothesis of most models. In [14] Batchelor investigated axisymmetric turbulence, which is still a strong constraint on the flow. For instance, for axisymmetric turbulence the Eulerian autocorrelation  $R_{ij}$  can be decomposed into four<sup>3</sup> dimensionless functions  $A$ ,  $B$ ,  $C$  and  $D$  depending

<sup>3</sup>Remember that in isotropic turbulence  $R_{ij}$  is fully described by  $f$  and  $\langle u_L \rangle$ .

on  $r$  and the ratio between the angular and radial velocity. Analogous to equation (III.47) the continuity equation dictates that  $A$ ,  $B$ ,  $C$  and  $D$  are related. However, this relation is also much more complicated than equation (III.47). Furthermore, it is possible to define four Taylor scales and four integral length scales. This complicates measurements and models further. Therefore, an isotropic flow is preferred in most models.

To investigate turbulence, wind tunnels and mixers were able to provide local isotropy and homogeneity to a good extent. However, in a wind tunnel the fluctuating velocity is typically 5% to 20% of the mean velocity. Many Lagrangian measurement techniques observe a small, fixed area. For example, particles in a wind tunnel sweeping through a  $(50\text{ cm})^3$  observation volume at  $5\frac{\text{m}}{\text{s}}$ , pass the volume in average in  $\frac{1}{10}\text{s}$ . If the fluctuating velocity is small compared to the mean velocity, then the measured Lagrangian trajectories are only few  $\tau_\eta$  (or even less) short. Therefore, the creation of mean velocities much smaller than the turbulent fluctuations is a major requirement for Lagrangian measurements.

On the other hand, mixers like the very common *counterrotating von Kármán machine*<sup>4</sup> i.e. a cylindrical tank with two counterrotating propellers at the top and the bottom, create less mean flow but an axisymmetric flow field with a very small approximately isotropic region in the center of the apparatus. The Lagrangian Exploration Module can be seen as six pairs of counterrotating von Kármán machines interacting in one apparatus. This arrangement provides a flow which is locally isotropic in a region of the size of the integral length scale and nearly negligible mean flow. More details on the setup of the machine can be found in chapter V.

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<sup>4</sup>This arrangement is also called French–Washing–Machine.



## IV Overview of measurement techniques

In hydrodynamics and aerodynamics a broad collection of measurement techniques had been developed. This chapter introduces some common techniques. Particle tracking velocimetry will be explained in detail since it was used to obtain the data.

### IV.1 Hot wire Anemometry

In hot wire anemometry a current is passed through a thin wire and thus the wire is heated to a temperature  $T$ . The resistance of the wire is related to its temperature. Gas passing by the hot wire convects heat away and thus changes temperature, which consequently changes the resistance. Either the temperature or the current is kept constant and by measuring the other parameter as a function of time one can obtain the velocity at the wire. With hot wire anemometry it is not possible to determine the flow direction. In addition this method can only be used in gas flows with a high mean flow.

### IV.2 Laser Doppler Velocimetry

Laser-Doppler-Velocimetry (LDV) allows the non-intrusive measurement of velocity and acceleration at a small measurement volume. As pictured in figure IV.1, a laser is split in two. The frequency of one of the two beams is shifted by  $\Delta f$  (typically several MHz) using Bragg-reflexion in a so called Bragg-cell. Thereafter the beams are aligned such that they cross at the region of interest. The cigar shaped region is typically a few millimeters long and a few tenth of a millimeter wide. Because of interference a pattern of parallel planes with high light intensity is created. The distance  $\Delta x$  between the so called fringes is:

$$\Delta x = \frac{\lambda}{2 \sin(\phi/2)} \quad (\text{IV.1})$$

where  $\lambda$  denotes the wave length of the laser and  $\phi$  the angle between the two beams. Because of the slight frequency shift the stripes are moving with velocity  $v_f = \Delta f \Delta x$ . Particles passing the measurement volume scatter modulated light. The modulation is the beat frequency of the fringes, which is Doppler shifted by the velocity component perpendicular to the fringes. A receiver, consisting of a photo multiplier and fast signal processor, measures the light intensity as a function of time. The velocity is calculated from the frequency of the signal. As described in [15] it is

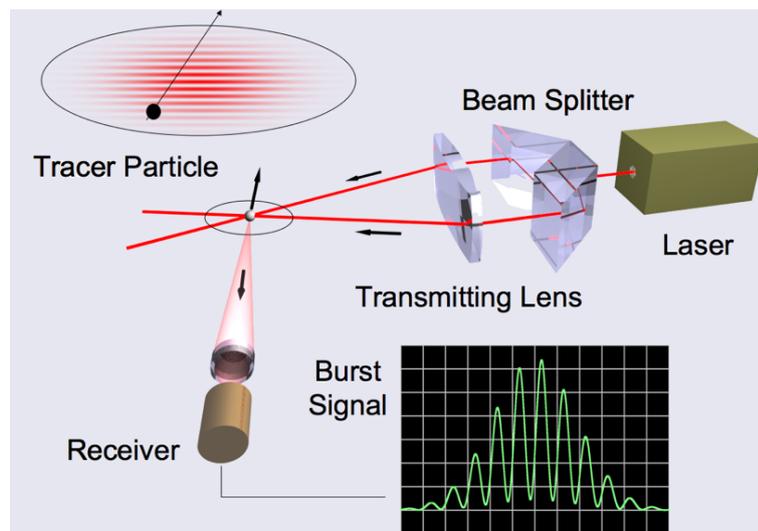


Figure IV.1: Sketch of a LDV system. Image courtesy of Holger Nobach. A laser beam is split in two parts, whereas one is frequency shifted in a Bragg cell; both beams cross at the measurement volume and interfere; because of the frequency shift the interference stripe pattern is moving with a velocity  $\mathbf{v}$ ; particles crossing the intersection, will blink with a frequency depending on their own velocity, the angle between the beams and the beat frequency; acceleration and velocity can be obtained from frequency of the beating signal

possible to obtain particle accelerations from laser-Doppler measurements using also the change in frequency of the time the particle passes through the measurement volume. Unfortunately, only one particle at a time can be measured. Using two or three LDV systems, more components of the velocity can be obtained.

### IV.3 Particle Image Velocimetry

Particle Image Velocimetry (PIV) (see e.g. the review by Adrian [16]) uses one camera to acquire a two dimensional velocity field. A thin laser sheet is directed to illuminate a region in a flow seeded with small particles. One (or more) camera(s) is orientated at the illuminated region taking two consequent images. The time difference between the images,  $\Delta t$ , is typically very small. Thus, either high speed film cameras or special PIV cameras are used. The first image is subdivided into small blocks and the cross correlation between each block and the image  $\Delta t$  later is calculated. The point with the maximum cross correlation is considered as the new position of the block. Thus, one can calculate the velocity of all blocks. Using more cameras it is possible to increase the accuracy and robustness of the measurement. Furthermore, two cameras observing from different angles enable to obtain three-dimensional velocity data on the plane defined by the thin laser sheet.

## IV.4 Tomographic Particle Image Velocimetry

Tomographic particle image velocimetry (tomo-PIV) is a recent development with which one can obtain three dimensional velocity fields. Three or four cameras are directed on an illuminated measurement volume and take synchronously images. By using tomographic methods two 3D density fields of the two image sets are created. Analogous to PIV, the cross correlation between a region from the first density field and the second field is calculated. The maximum in the cross correlation is considered to be the new position of the selected region. The velocity field can be obtained from the three-dimensional displacement of each block from  $t$  to  $t + \Delta t$ .

## IV.5 Particle Tracking Velocimetry

Most measurement techniques (e.g. Hot wire Anemometry, PIV and LDV ) allow observation from the Eulerian point of view, meaning that the velocity is measured at one – or several – fixed points in space. Measurement techniques working in the Lagrangian framework make it possible to follow fluid segments along their trajectories in the flow using particles as a probe.

In principle, Eulerian and Lagrangian measurements can be transformed into their counterpart. In the case of two-dimensional turbulent flows the transformation can be realized from PIV measurements [17, 18]. Yet, transforming three-dimensional Eulerian to Lagrangian fields (and vice versa) is still impossible due to the extremely high requirements on resolving the flow. Therefore, the Lagrangian framework is more convenient to study Lagrangian correlations and structure functions, as well as particle dispersion and mixing.

The basic idea of Particle Tracking Velocimetry<sup>1</sup> (PTV) is to determine the three dimensional position of tracer particles in the flow at certain times and to build up the points into trajectories of particles. Afterwards the velocity and acceleration can be computed from the trajectories. Two ways to detect a tracer particle have been created so far: using the Doppler shift of a scattered wave, and optical detection using digital cameras. The scattered wave can either be created through ultrasound [19] or a modified LDV system [20].

We are using an optical method with a calibration method proposed by Tsai [21].

### IV.5.1 Experimental Setup

A laser beam or a bright light source illuminates particles in a measurement volume. The scattered light from a particle is detected by two or more high speed cameras looking from different angles into the volume. In other words particles produce an image on the image sensors of each camera. After taking synchronized movies, these are downloaded to a computer for the evaluation process.

The PTV method used in the experiment consists of the following three steps, which are: segmentation (IV.5.4), stereo matching (IV.5.5) and connection of the

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<sup>1</sup>also called Lagrangian Particle Tracking (LPT)

points to trajectories (IV.5.6). However, before a PTV system can be used it has to be calibrated according to the following camera model and calibration procedure.

## IV.5.2 The Camera Model

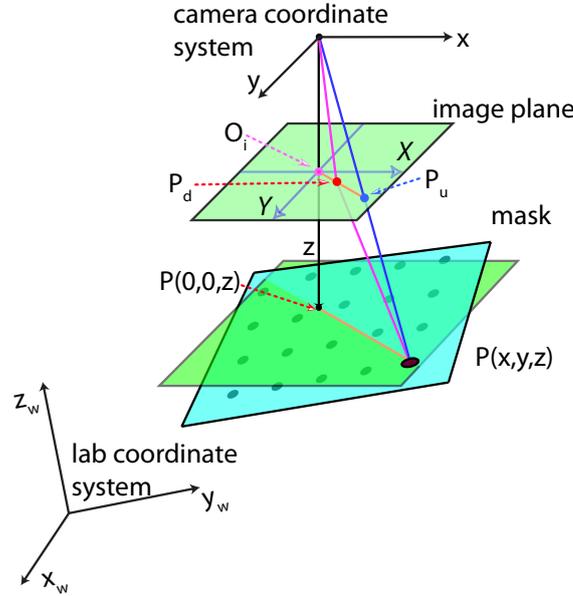


Figure IV.2: Sketch of the camera model proposed by Tsai [21]. The two green planes are parallel. For calibration a mask (blue) with a known dot pattern is moved along the  $z_w$ -axis. It is assumed that the image is only distorted along the orange lines. This is the radial alignment constraint used in the calibration.

As proposed by Tsai [21], modern off-the-shelf cameras and lenses are well described by a pin hole camera with radial distortion but negligible tangential distortion.

The mathematical projection of a point in 3D onto an image sensor of a model camera can be done in four steps. A sketch of the different coordinate systems is provided in figure IV.2.

**Step 1:** A point  $\begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$  in the object world coordinate system can be transformed to the camera coordinate system by rigid body transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\underline{\mathbf{R}}} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \mathbf{T} \quad (\text{IV.2})$$

$\underline{\underline{\mathbf{R}}} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}$  is a  $3 \times 3$  rotation matrix and  $\mathbf{T} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$  a translation vector. Since  $\underline{\underline{\mathbf{R}}}$  can be expressed in terms of three Eulerian angles and the

translation also has three degrees of freedom, this step involves six unknown parameters.

**Step 2:** Transformation of the 3D camera coordinates to ideal (undistorted) image coordinates  $(X_u, Y_u)$ .

$$X_u = f \frac{x}{z} \quad Y_u = f \frac{y}{z} \quad (\text{IV.3})$$

Although the focal length  $f$  should be known from the camera's specifications, it is used a free fitting parameter during calibration. In fitting  $f$  possible distortions caused by the windows of the apparatus can be included and better accuracy can be achieved.

**Step 3:** Only radial distortion is considered to be relevant for modern lenses. The correction can be expressed as

$$X_d + D_x = X_u \quad Y_d + D_y = Y_u \quad (\text{IV.4})$$

with

$$D_x = X_d (\kappa_1 r^2 + \kappa_2 r^4 + \dots) \quad (\text{IV.5a})$$

$$D_y = Y_d (\kappa_1 r^2 + \kappa_2 r^4 + \dots) \quad (\text{IV.5b})$$

$$r = \sqrt{X_d^2 + Y_d^2} \quad (\text{IV.5c})$$

Carrying out the correction only up to the first order provides good results at low computational costs and is numerically stable. The radial distortion factor  $\kappa_1$  is introduced as another unknown parameter to be determined during calibration.

**Step 4:** The real image coordinate has to be transformed to the pixel representation in the computer:

$$X = X_f - C_x = \frac{s_x X_d}{d_x} \quad (\text{IV.6a})$$

$$Y = Y_f - C_y = \frac{Y_d}{d_y} \quad (\text{IV.6b})$$

with  $(X_f, Y_f)$  the position in pixels,  $(C_x, C_y)$  the position of the image center (in pixels),  $d_x$  and  $d_y$  the size of a pixel in  $x$  and  $y$  direction respectively and the so-called uncertainty image scale factor  $s_x$ . The latter allows to correct hardware issues of the cameras. In the case of an ideal sensor  $s_x$  is unity. However, it is kept as the ninth unknown parameter to be identified by the calibration.

The center of the image plane  $(C_x, C_y)$  as well as the size of a pixel  $(d_x, d_y)$  is assumed to be known.

All these steps need detailed information about the lab coordinate system, cameras, etc. The nine parameters can be determined using a calibration method described in the following subsection.

### IV.5.3 Calibration

The four steps described in subsection IV.5.2 involve nine unknown parameters depending on the arrangement of the cameras, the cameras itself and the objectives. The calibration method proposed by Tsai [21] allows to determine these parameters in a robust and accurate way.

First, one has to take images of a known structure, such as a regular dot pattern with a marked center, at several known positions in the lab system. To do so one usually moves a plate<sup>2</sup> with a regular dot pattern along an axis in the lab system. To be compliant with figure IV.2, we will assume that the mask is moved in  $z_w$ -direction.

In a second step, the images are being processed such that one gets a set of 3D points  $(x_w, y_w, z_w)_i$  in the lab system and their corresponding projection on the image plane  $(X_i, Y_i)$ . If only radial distortion of the image is considered, we can always choose the rotation matrix  $\underline{\mathbf{R}}$  and  $\mathbf{T}$  such that  $\underline{\mathbf{O}}_i \mathbf{P}_d$  is parallel to  $\underline{\mathbf{P}}(0, 0, z) \mathbf{P}(x, y, z)$  (as displayed by the orange lines in figure IV.2). This can be rewritten as:

$$\begin{aligned} (X_d, Y_d) \times (x, y) &= 0 \\ \Leftrightarrow X_d \cdot y &= Y_d \cdot x \end{aligned} \quad (\text{IV.7})$$

Inserting equation (IV.2) into equation (IV.7) yields:

$$X_d (r_4 x_w + r_5 y_w + r_6 z_w + T_y) = Y_d (r_1 x_w + r_2 y_w + r_3 z_w + T_x) \quad (\text{IV.8})$$

This constraint is called *radial alignment constraint*. Rearranging equation (IV.8), one can see that it yields a set of linear equations with seven unknowns.

$$\begin{bmatrix} Y_d x_w & Y_d y_w & Y_d z_w & Y_d & -X_d x_w & -X_d y_w & -X_d z_w \end{bmatrix} \cdot \mathbf{A} = X_d \quad (\text{IV.9})$$

where  $\mathbf{A} = \begin{bmatrix} T_y^{-1} s_x r_1 & T_y^{-1} s_x r_2 & T_y^{-1} s_x r_3 & T_y^{-1} s_x T_x & T_y^{-1} r_4 & T_y^{-1} r_5 & T_y^{-1} r_6 \end{bmatrix}^T$  contains the unknown parameters.

If the number of 3D points found is much higher than seven, equation (IV.9) is overdetermined. In this case one can find  $\mathbf{A}$  with a least square fit. For any three dimensional rotation matrix  $\underline{\mathbf{R}}$  it is  $\sum_{i=1}^3 R_{ij}^2 = \sum_{j=1}^3 R_{ij}^2 = 1$ . Thus one can determine  $|T_y|$ ,  $r_7$ ,  $r_8$ ,  $r_9$  and  $s_x$  analytically.

The last three steps in the camera model can be combined to obtain the equations

$$f \frac{x}{z} = X_u = X \frac{d_x}{s_x} (1 + \kappa_1 r^2) \quad (\text{IV.10a})$$

$$f \frac{y}{z} = Y_u = Y d_y (1 + \kappa_1 r^2) \quad (\text{IV.10b})$$

Expressing  $x$ ,  $y$  and  $z$  in terms of  $x_w$ ,  $y_w$  and  $z_w$  yields:

$$f \frac{r_1 x_w + r_2 y_w + r_3 z_w + T_x}{r_7 x_w + r_8 y_w + r_9 z_w + T_z} = X \frac{d_x}{s_x} (1 + \kappa_1 r^2) \quad (\text{IV.11a})$$

$$f \frac{r_3 x_w + r_5 y_w + r_6 z_w + T_y}{r_7 x_w + r_8 y_w + r_9 z_w + T_z} = Y d_y (1 + \kappa_1 r^2) \quad (\text{IV.11b})$$

<sup>2</sup>Also called mask

Setting  $\kappa_1$  to zero in a first run, one gains an overdetermined set of linear equations, which enables us to acquire good initial guesses for  $f$  and  $T_z$ . The precise values of  $\kappa_1$ ,  $f$  and  $T_z$  can be found by employing a standard fitting scheme to equation (IV.11b) and the found 3D points. This calibration method allows an accurate and robust determination of the nine independent parameters in the camera model. It is at least as accurate as and more robust than optimization schemes working on all of the nine parameters at the same time.

#### IV.5.4 Particle Finding

The scattered light from the particle leaves a bright spot on the digital image. Such a spot has a size of several pixels and – in the case of spherical particles – can be represented by a two-dimensional Gaussian with added noise. The spots are identified using a segmentation technique which considers every pixel with an intensity above a certain threshold as part of a spot.

Ouellette *et al* [22] discussed several methods to identify the center of a spot and tested these methods on their performance and accuracy. The discussed attempts are

**two-dimensional Gaussian:** Homogenous spherical particles should – to a first approach – result in an intensity profile which can be described as a two-dimensional Gaussian. However, the fitting process is computationally expensive.

**two one-dimensional Gaussians:** Fitting a one dimensional Gaussian in x direction and one Gaussian in y direction produces mostly the same accuracy as the two-dimensional Gaussian but the computational cost of the fitting are minimal since they can be calculated from an analytical solution.

**Center-Of-Mass:** Non-spherical particles and particles<sup>3</sup> which are not uniform, will most likely produce a non-elliptical shape with several intensity peaks. In this case, the Center-Of-Mass can compute the approximate center of the spot with greater accuracy than a Gaussian approach.

**Neural Networks:** After training, also neural networks achieved a good accuracy even for noisy images.

All the experiments described with in this diploma thesis have been performed using spherical particles. Therefore, the two one-dimensional Gaussian method was sufficient.

#### IV.5.5 Stereo-matching

After finding the center of a light spot, a line of sight can be calculated by carrying out the steps described in IV.5.2 in reverse order. A point from the first camera is picked, and the calculated line of sight is projected with some tolerance band

<sup>3</sup>For example: biological particles, such as algae and copepods

(typically three pixel wide) on the other cameras. All spots within this band are now projected in 3D and the minimum distance between for each combination of lines of sight is calculated. If the minimum distance is below a threshold, it is considered as a real point. The stereo-matching process is done for each frame.

### IV.5.6 Building the trajectories

The points in 3D space at different timesteps are yet not associated with each other. The connection of the points, obtained from the stereo-matching step, into a trajectory,  $\tilde{\mathbf{x}}(t)$ , in time is called tracking. In [22], Ouellette *et al* investigated the following techniques on their accuracy:

**Nearest Neighbor:** Starting with a particle  $\tilde{\mathbf{x}}(t)$  every point  $\tilde{\mathbf{x}}(t + \Delta t)$  on the next frame within a maximum distance  $\|\tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t + \Delta t)\| < d_{max}$  can be considered a possible part of the trajectory. In order to detect high velocities,  $d_{max}$  has to be large enough, and because of having several particles in the measurement volume multiple continuations of a track are usually possible. Hence, it is recommended to look several frames ahead to avoid building physically meaningless trajectories.

**Minimal Acceleration – 3 Frames:** Using the particle position at  $t - \Delta t$  and the one at the current frame, the velocity can be calculated and the position at  $\tilde{\mathbf{x}}_p(t + 1 \Delta t)$  can be predicted. For all particles in the search volume around the estimate the acceleration can be calculated. The trajectory showing minimal acceleration is considered to be more likely to be physically correct.

**Minimal Change in Acceleration – 4 Frames:** The position, velocity,  $\tilde{\mathbf{v}}_p$ , and acceleration,  $\tilde{\mathbf{a}}_p$ , of the particle in frame  $n + 1$  is estimated using the described "Minimal Acceleration" method (as described above). The position in frame  $n + 2$  can be estimated as  $\tilde{\mathbf{x}}_p(t + 2\Delta t) = \tilde{\mathbf{x}}(t) + \tilde{\mathbf{v}}_p \cdot (2\Delta t) + \tilde{\mathbf{a}}_p \cdot (2\Delta t)$ . For all points in a search region around  $\tilde{\mathbf{x}}_p(t + 2\Delta t)$  the third time derivative, (i.e. change in acceleration), is calculated. The best match for the frame  $n + 1$  is the particle showing the minimum change in acceleration from the prediction in frame  $n + 2$ .

**Best Estimate – 4 Frames:** Instead of the third time derivative for frame  $n + 2$ , the distance to the estimate  $\tilde{\mathbf{x}}_p(t + 2\Delta t)$  is calculated for all points in the search volume. The best match for the frame  $n + 1$  is now the particle showing minimum distance between estimate  $\tilde{\mathbf{x}}_p(t + 2\Delta t)$  and points  $\tilde{\mathbf{x}}_i(t + 2\Delta t)$ .

At the beginning of a trajectory a *Nearest Neighbor* has to be used. Ouellette *et al* [22] showed that the *Best Estimate* produces the best results even for higher particle densities. Therefore, this method had been implemented in the particle tracking method.

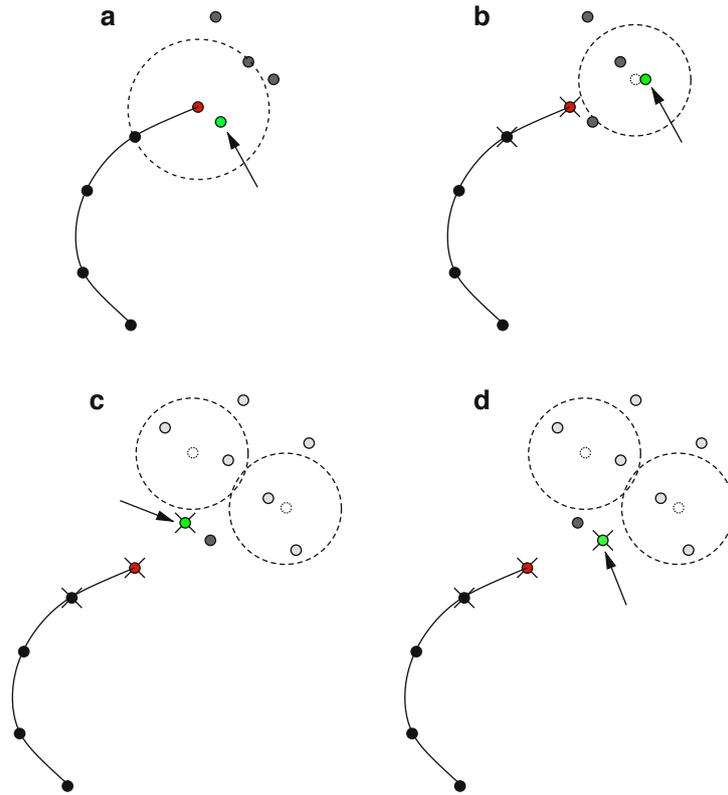


Figure IV.3: Sketch of the discussed techniques to build the trajectories. Image taken from [22]. Red dots indicate the particle position at  $t$ , and the arrows point at the green particle chosen for  $t + \Delta t$ . The sub figures are: *a: Nearest Neighbor* As indicated by the arrow, the particle at  $t + \Delta t$  which is closest to the particle at  $t$  was picked. Clearly, this method has a high risk of building physically incorrect trajectories. *b: Minimal Acceleration* Using the already known particle positions at  $t$  and  $t - \Delta t$  one can estimate the position in frame  $t + \Delta t$ . The particle closest to point will be picked. *c: Minimal Change in Acceleration* By looking one frame back and one frame ahead one can estimate velocity and acceleration for all possible continuations of the track. The particle at  $t + \Delta t$  for which the particle in frame  $t + 2\Delta t$  shows the least change in acceleration, will be chosen. *d: Best Estimate* Instead of minimizing the change of acceleration, the particle at  $t + \Delta t$  is picked which shows the best estimate for a particle in  $t + 2\Delta t$ . The PTV system employed in this diploma thesis uses the *Best Estimate* technique.

### IV.5.7 Connection of Tracks

Fluctuations in the intensity of the illumination, and hardware (e.g. dead pixels) and software problems can break up trajectories.

To avoid this H. Xu [23] developed a technique to reconnect trajectories broken into several shorter pieces using linear interpolation in a six dimensional position–velocity space. The algorithm is explained in figure IV.4. The reconnected trajectories are much longer in duration, thus providing much better Lagrangian statistics. However, the interpolated points themselves can not be used since they would introduce an artificial bias to the statistics.

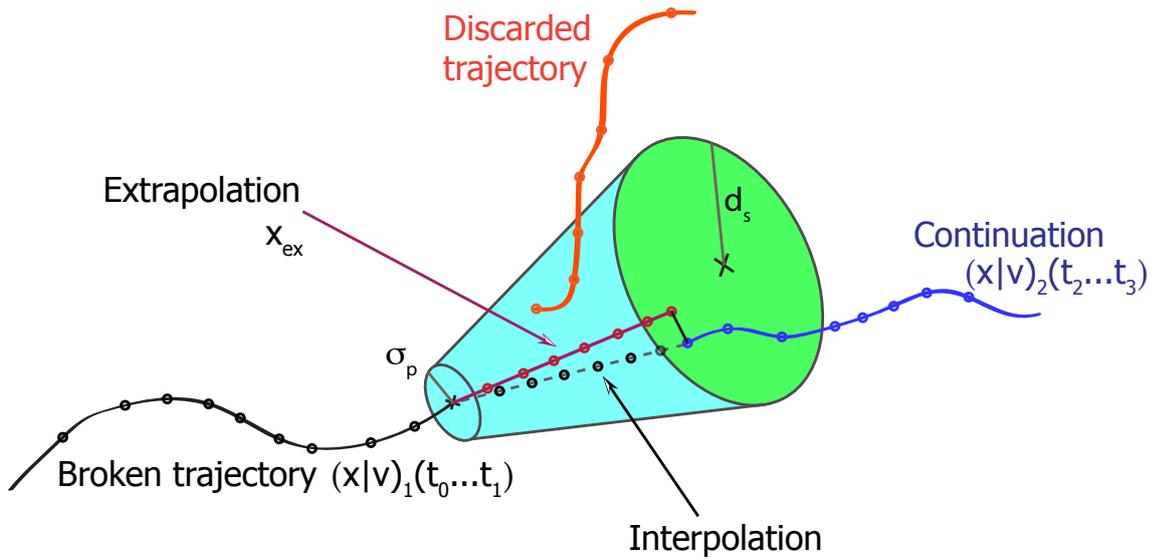


Figure IV.4: The broken trajectory  $(\mathbf{x}, \mathbf{v})_1(t)$  is extrapolated from its end,  $t_1$ , up to temporal search distance,  $t_{max}$ , along its last known velocity vector. Trajectory starts at time  $t_2$ , found within a spatial search distance  $d(t) = \sigma_p + t/t_{max} d_s$  to the extrapolation, are possible candidates for a continuation of the track.  $\sigma_p$  is the uncertainty in position measurement and  $d_s$  is user set maximum. The search area can be imagined as a truncated cone. Nevertheless, the cone is situated in four-dimensional time-position space. For each candidate an extrapolation  $\mathbf{x}_{ex}(t_2) = \mathbf{x}_1(t_1) + (t_2 - t_1) \frac{\mathbf{v}_1(t_1) + \mathbf{v}_2(t_2)}{2}$  and its distance to the candidate are calculated. The trajectory showing the least distance is picked as the continuation. The unknown points between  $\mathbf{x}_1(t_1)$  and  $\mathbf{x}_2(t_2)$  are created through a linear interpolation. For the linear interpolation the direct connection between  $x_1(t_1)$  and  $x_2(t_2)$  is established. To avoid a bias of the data, these artificial points are marked as interpolated.

# V Experimental Setup

## V.1 The Lagrangian Exploration Module

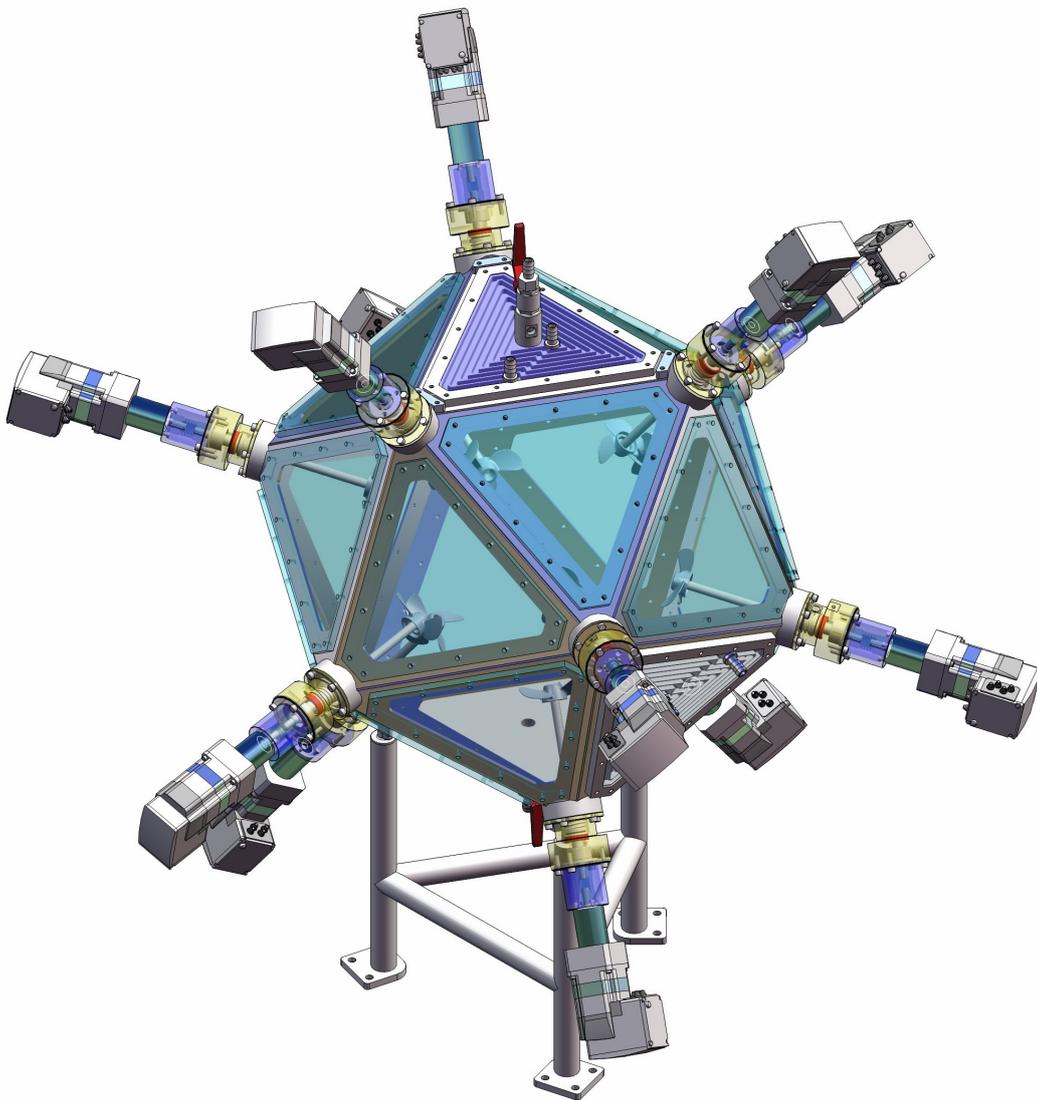


Figure V.1: CAD drawing of the LEM

### V.1.1 Basic structure

The Lagrangian Exploration Module (LEM) has a skeletal structure made of stainless steel in the shape of an icosahedron. The edge's length is 40cm — leading to a total volume of 140 liters. On each face either a plexiglas window or a cooling plate can be mounted. To avoid corrosion, all parts that are in contact with water, are either made of plastic or stainless steel. The top and the bottom plate are two cooling plates which have an additional hose attached to them, so that the tank can be filled with, or emptied of, the working fluid. On each of the 12 vertexes a propeller unit is mounted.

15 M6 screws per face are used to mount windows and cooling plates. Each screw is screwed in from the inside of the tank with a Nylon washer between the screw head and the LEM skeleton. From the outside, standard stainless steel nuts and washers are used to fix the plate. Additionally, an o-ring is placed along the edge between the triangular window or cooling plate and the LEM structure to avoid leakage. Screw glue is added to help seal off the screw threads. Applying high vacuum grease to the o-ring also proved to be an effective way to prevent leakages.

### V.1.2 Propeller unit

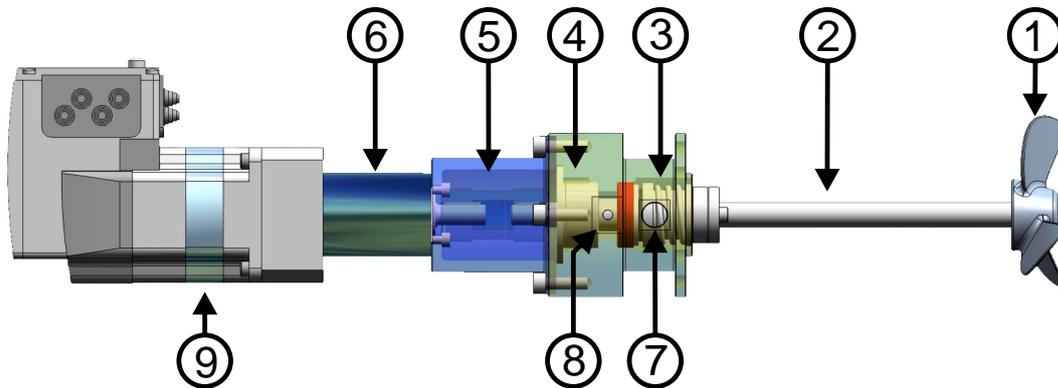


Figure V.2: CAD drawing of the propeller unit; 1: propeller, 2: exchangeable propeller shaft, 3: rotating seal, 4: bearing, 5: shaft coupling, 6: gearbox, 7: threaded hole connecting the propeller unit to the bubble trap, 8: hole allowing water vapor produced by the seal to leave the housing, 9: brushless DC motor

A propeller unit (PU) is mounted to each vortex. Figure V.2 shows a CAD drawing of the propeller unit. Both propeller and propeller shaft can be replaced, allowing different propellers to be tested at different distances into the tank. The inner part of the shaft is held by a standard double-row ball bearing. Sealing is achieved with a rotating seal (Pac Seal). The propeller is driven by a brushless DC-motor (IFE71, Berger-Lahr) whose speed is reduced by a 5 : 1 planetary gear box. The gearbox and inner shaft are connected by a stainless steel flexible coupling. A threaded hole in front of the rotating seal makes it possible to connect a hose to

the propeller unit, which can be used for particle insertion or bubble removal. A second hole drilled on the other side of the seal allows water vapor produced by the rotating seal to leave the housing.

## Motors

Each motor is supplied with a constant voltage of 36 V and connected to a CANbus. A maximum power of 120 W can be conferred to the flow. The power is provided by two adjustable 1.5 kW power supplies (N5766A, Agilent). It is possible to read out the actual voltage and current of both power supplies via ethernet.

Via CANopen commands sent over the CANbus, speed and acceleration can be set (and changed during runtime) for each individual motor. The maximum speed is  $\pm 5000$  rpm (resp.  $16\frac{2}{3}$  Hz after the 5 : 1 gear box) and the minimum speed, apart from 0 rpm, is  $\pm 300$  rpm (1 Hz after gear box). The acceleration can be set between 1000 rpm/s and 10000 rpm/s. If prompted, a motor will return its actual status, displaying for example its temperature, speed and current.

A PC with a CAN-USB adapter (PCAN-USB, isolated – Peak Systems GmbH) is used to control the motors. The software is written in Delphi providing an automatic and a manual mode. In the automatic mode a text file, specifying the speed and rotation sense for each individual motor at different times, is interpreted by the program.

## Propellers

In a first attempt, we used round anodized aluminum disks of 10 cm diameter cut in at every 60 degrees, where the corners were bent in (see figure V.3). As the first trial produced a highly rotating flow with little pushing (measured by hand), we bent the vanes some more until they stood out at an angle of about 30 degrees. The result was satisfactory. It should be noted that the propellers were bent by hand, so that each of them produces a slightly different flow.



Figure V.3: figure on the left: propeller made out of anodized aluminum, the diameter is 10 cm;

figure in the middle: a propeller after 1 week in saline water (15 g marine salt per liter) – the dent has a depth of  $\approx 2$  mm

figure on the right: stainless steel propeller of the same size and shape

Although anodized, the propellers were still subject to corrosion in salt water (see figure V.3). Consequently, they were replaced by stainless steel propellers of the same size and approximately the same shape. In order to test the influence of different propeller sizes on the flow we further designed a four-vane plastic propeller with exchangeable vanes.



Figure V.4: plastic propeller: up to 4 vanes of different sizes can be mounted

### V.1.3 Particle Insertion & Bubble Removal

The removal of bubbles and insertion of particles is done by the same device – the bubble trap which is shown in figure V.5. For that purpose all downward pointing propeller units (PU) are connected to the upper inlet of a container – a modified filter housing (Thermo Fisher Scientific). The lower outlet in return is linked to the upward pointing PUs through a pump. The content of the container is pumped through the six upward pointing PUs into the LEM. Hence, the fluid and bubbles which are trapped in these PUs are transported back into the container. There, air bubbles can escape due to their buoyancy. On the other hand, new particles inserted into the container, will be pumped into the LEM.

It is possible to connect the container to a vacuum pump to lower the pressure in the tank. In this case, it is necessary to bypass the pump using the three valves shown in figure V.5. Assuming a horizontal throw of water out of the inlet of the container a flow rate through the pump of about  $1 \frac{l}{min}$  was calculated.

### V.1.4 Filtering & Plumbing

We are using two filter housings with a  $10 \mu m$  and a  $5 \mu m$  filter. Both were bought at Thermo Fisher Scientific. A garden pump from the local Bahr Baumarkt is used to pump water from the connector at the base plate of the LEM through the coarse and the fine filter and back into the LEM through the top plate.

The connectors, plugs, distributors and valves integrated into the water system were manufactured by Gardena. The standard hose also belonging to the water system has a diameter of half an inch.

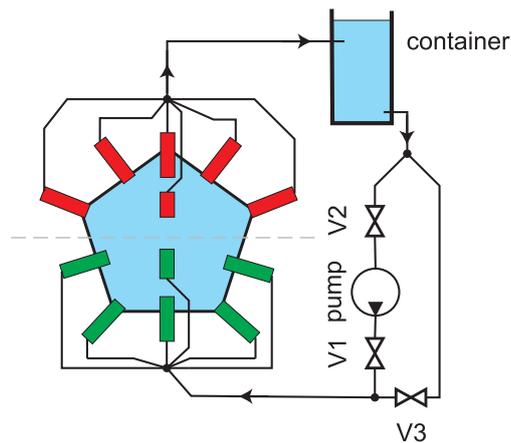


Figure V.5: Particle insertion and bubble removing: particles inserted into the container will be pumped into the LEM through the six upper propeller units (green); on the other hand, bubbles which get trapped in one of the six lower propeller units (red) are pumped into the container, where they leave the system due to their buoyancy; the 3 valves (V1, V2 & V3) allow the bypass of the pump, which is required for depressurization of the tank.

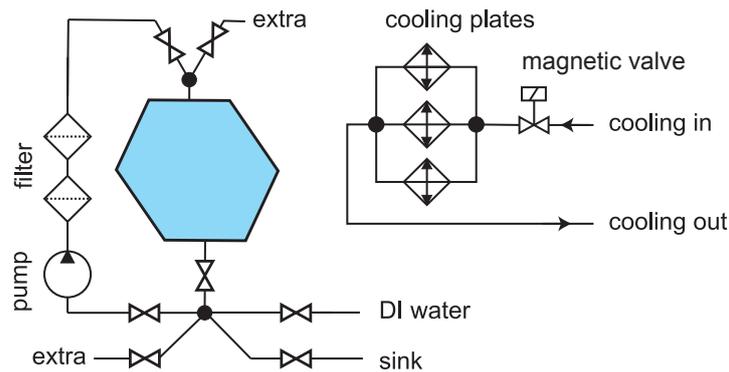


Figure V.6: Left side: schematic of the plumbing system used to fill and filter the tank: the LEM is filled with deionized water; through an extra inlet at the top and an extra outlet at the bottom samples can be taken or bigger amounts of chemicals can be administered to the tank; a pump can transfer the tank content through a  $5\ \mu\text{m}$  and a  $10\ \mu\text{m}$  filter to remove particles and dirt;

right side: schematic of the cooling system: in opening the magnetic valve the cooling water can run through the cooling plates mounted to the LEM; the micro-controller described in section V.1.5 opens the valve after comparing the measured temperature to a user-specified threshold.

### V.1.5 Temperature control

Without the cooling system heat would be a major problem of the LEM. Since the motors are connected to body of the LEM without any insulator in between, electronic and mechanical heat will raise the temperature of the fluid. For example,  $2\ \text{kW}$  will increase the temperature of  $140\ \text{l}$  of water by  $12\ \text{K}$  in one hour. To control

the temperature in the tank, we glued each a PT100<sup>1</sup> resistor to each of the top and bottom plates with silicone. As shown in figure V.7 both are separated by  $\approx 4\text{mm}$  thick layer of silicone from the metal surface. The bias introduced by attaching the sensors to the cooling plates is surpassed by the errors of the temperature measuring system, and therefore negligible.



Figure V.7: The PT100 sensor glued to a cooling plate; the wire is passed trough a former water outlet.

With the help of the circuit shown in figure V.8, the resistance of the PT100 is converted into a voltage signal. The circuit was adjusted and calibrated such that a range between  $0^\circ\text{C}$  and  $50^\circ\text{C}$  can be covered. The result is passed to a 10-bit analogue-digital converter of a micro-controller (C-Control 2, Conrad). If the moving average of the temperature over 2 seconds is higher than a user-set value, it opens a magnetic valve to flush the cooling plates with cooling water. The accuracy is  $0.4\text{K}$  due to the internal noise of the AD-converter of the micro-controller.

### V.1.6 PTV setup

The Particle Tracking Velocimetry system consists of 3 cameras, a laser, a control-PC and a cluster of several node computers, where the particle tracking program runs.

Figure V.9 shows a sketch of the wiring for the PTV system. Three Phantom V7.2 high speed cameras from Vision Research take 8-bit grayscale movies at frame rates up to  $11.000\text{ fps}$  at a resolution of  $512 \times 512$  pixels. The control and download of the movies is done via ethernet. A frequency generator creates a square wave, which is passed to a frequency divider. There, the frequency is adjusted to the frame rate of the three cameras. Then it is passed as a synchronization signal to the latter. A part of the unreduced output from the frequency generator is also passed to a delay unit. The delayed signal is then passed to the Q-switch of the laser. The cameras need a few micro seconds to prepare the electronics before a picture can be taken. Delaying the laser pulse allows the efficient capture of laser pulses for one

<sup>1</sup>A PT100 is thin platinum resistor with a resistance of  $100\ \Omega$  at  $0^\circ\text{C}$  and a linear temperature dependence of their resistance within 0 and  $100^\circ\text{C}$

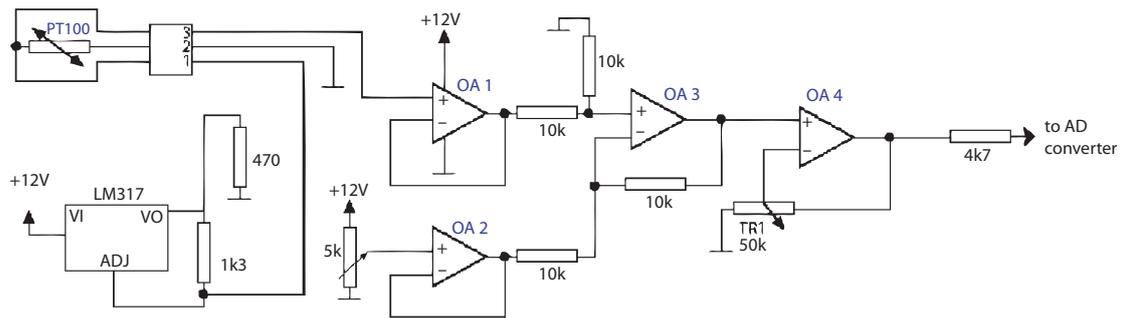


Figure V.8: Represented is the circuit converting the resistance of a PT100 to a voltage signal readable by an AD converter:

IC LM317 creates a constant current through the PT100; the voltage drop over PT100 is amplified 1 : 1 with an operational amplifier (OA) 1 to reduce noise; a stable offset voltage is created at OA 2; OA 3 subtracts the signal of OA 2 from OA 1 which corresponds to the subtraction of an offset temperature (a PT100 already has a resistance of  $100\ \Omega$  at  $0^\circ\text{C}$ ); in the last step OA 4 amplifies the input such that the maximum signal to be expected corresponds to the maximum input voltage of the AD converter.

frame. A small reed relay driven by the parallel port allows both to switch on or off the laser and to externally trigger the cameras. All cameras, the PC and the cluster are connected to a Gigabit ethernet switch.

Setting frame rate, resolution and other properties is done with a program (named `QTalkCam`) running on the control-PC. For the simultaneous recording of three movies, the same program switches on the laser via the relay and starts the recording (albeit at slightly different times) for each camera. Due to initial instabilities of the laser, heat up procedures, etc, the recording goes over a longer period than necessary for the specified amount of frames. The recording is stopped synchronously by switching off the laser. After terminating the recording process only a specified amount of the last frames is downloaded:

A server program (called `ph7server5`) running on the cluster is waiting to download the movies using the second network interface of the cameras. If the recording was successful, the control program `QTalkCam` transfers IPs, basic camera properties, the number of frames to download and a black-reference from each camera to the server program, which starts the simultaneous download of the movies. Once the download of all three movies has finished, `ph7server5` signals to the control program that the next movie set can be started. A third program called `controller_node` processes available movie sets according to the PTV algorithm described in section IV.5 with user specified thresholds. The original movies are usually deleted after computing and storing the trajectory files.

## Camera mounting

The challenge for developing a camera mounting system was to find a design compatible with the non-perpendicular angles between the faces of the LEM. The system

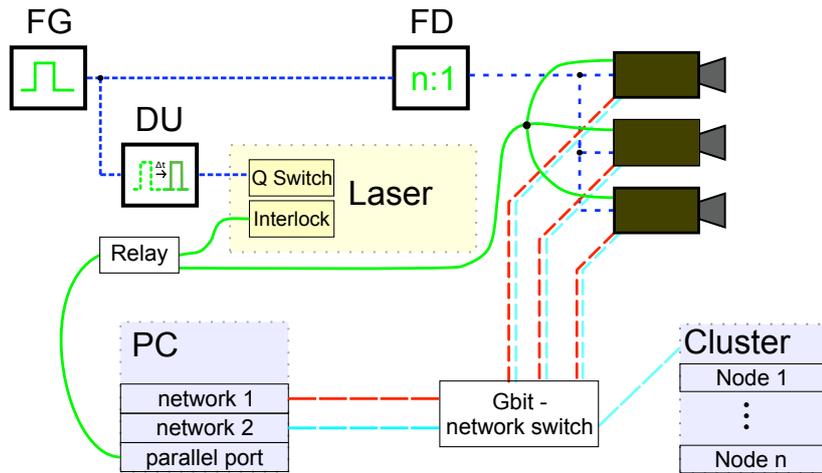


Figure V.9: Schematic wiring of the PTV system: The frequency generator (FG) generates a square wave (blue) with frequency  $> 10\text{kHz}$ , the signal is passed to a frequency divider (FD) and divided such that the new frequency corresponds to the frame rate of the cameras, the divided signal is the synchronization signal for the cameras; a delay unit (DU) delays the frequency such that laser pulses occur only after camera electronics are prepared; cameras are controlled via a PC (red), whereas movie downloading and particle tracking is done on a cluster connected to a second ethernet card on the cameras (cyan); operating the laser and triggering the cameras is done via a parallel port driven relay

was required to be easily adjustable, yet rigid enough to enable particle tracking velocimetry without additional position errors. We settled for Linos X95 profiles, and mounted them with adaptor plates on a platform. The platform consists of two  $600\text{ mm} \times 450\text{ mm}$  aluminum plates with a M6-threaded  $25\text{ mm}$  hole pattern from Thorlabs mounted on a frame built out of  $45\text{ mm}$  aluminum profiles (FM-Systeme).

The cameras are attached to the Linos profiles with the help adaptors built by our machine shop. The camera mounts for the two side-cameras were constructed in such a way that a rotation will not affect the position of the CMOS-chip. Figure V.10 shows the camera mount for the top camera. The design enables the camera to be both tilted and moved in a range of about  $3\text{ cm}$  in the direction towards the LEM.

Figure V.11 shows two pictures of the camera arrangement. After calibration, the camera position is fixed. Any unintentional movement can be registered by a Matlab code written by Haitao Xu. This program measures the shift and rotation of the image plane with respect to its original position during calibration. It was observed that the shift after a few weeks was less than a pixel. No rotation of the image plane could be measured. Therefore, the structure has proven its stability.

## Light

Light is provided by a frequency doubled Nd-YAG laser (DDC technologies) with a high pulse rate. Its power during the experiments was approximately  $30\text{ W}$  at pulse rates between  $10\text{ kHz}$  and  $30\text{ kHz}$ . Each frame captured several laser pulses.

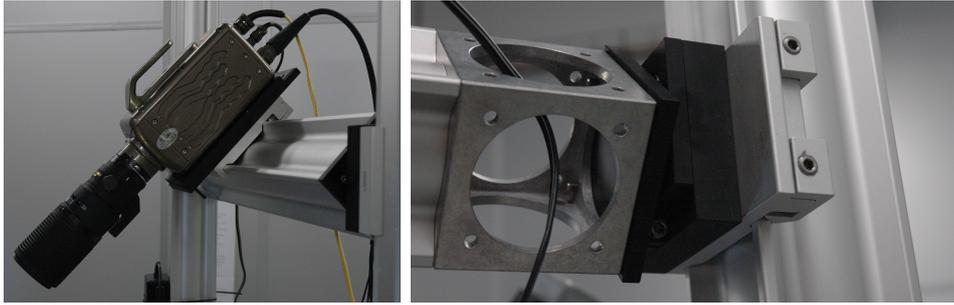


Figure V.10: Camera mounts for the top camera: figure on the left: height and position in  $y$ -direction can be changed with the Linos carriers; the figure on the right: the piece which allows tilting of the camera.

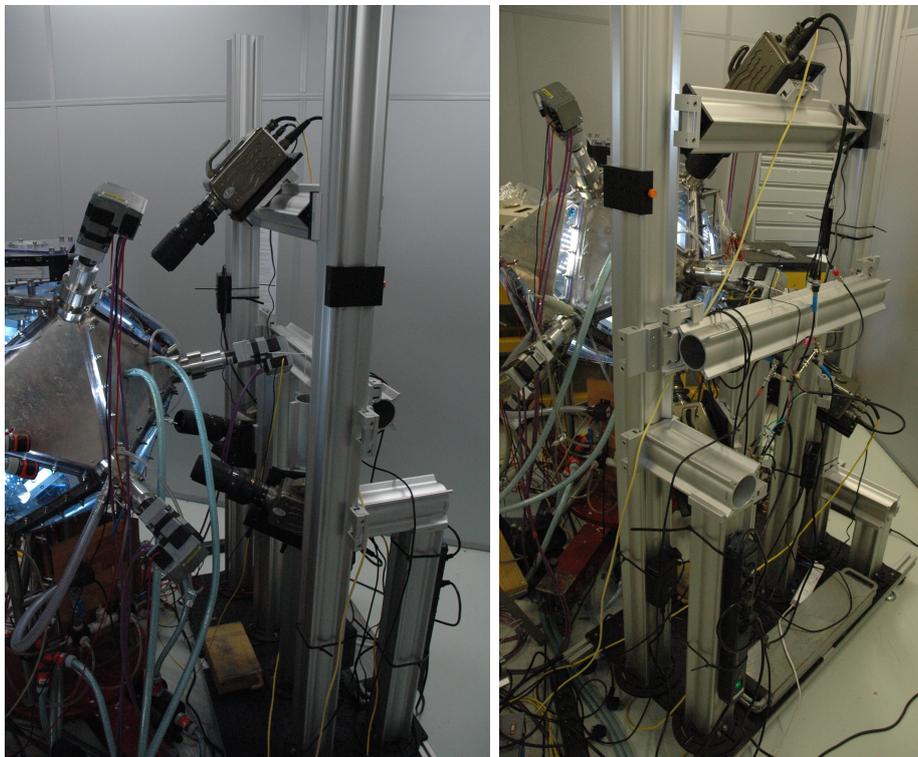


Figure V.11: figure on the left: side view from the construction holding the cameras; figure on the right: backside view from the structure on which the cameras are mounted.

The beam was expanded via a convex and a concave lens mounted on a tilted Linos X95 profile. The whole system was aligned in a way that the beam enters the LEM perpendicular to a face and passes through the center of the volume. For security reasons the face on the opposite side has to be blocked. Therefore, we mounted an additional cooling plate onto this site.

Initially, 12 high power LEDs (Ostar HEX-6,  $\approx 1000 \text{ lm}$  of white light, Osram) and 18 less powerful LEDs (Osram) were added to the setup to increase the amount of light in the measurement volume. However, the attempt was abandoned, since a

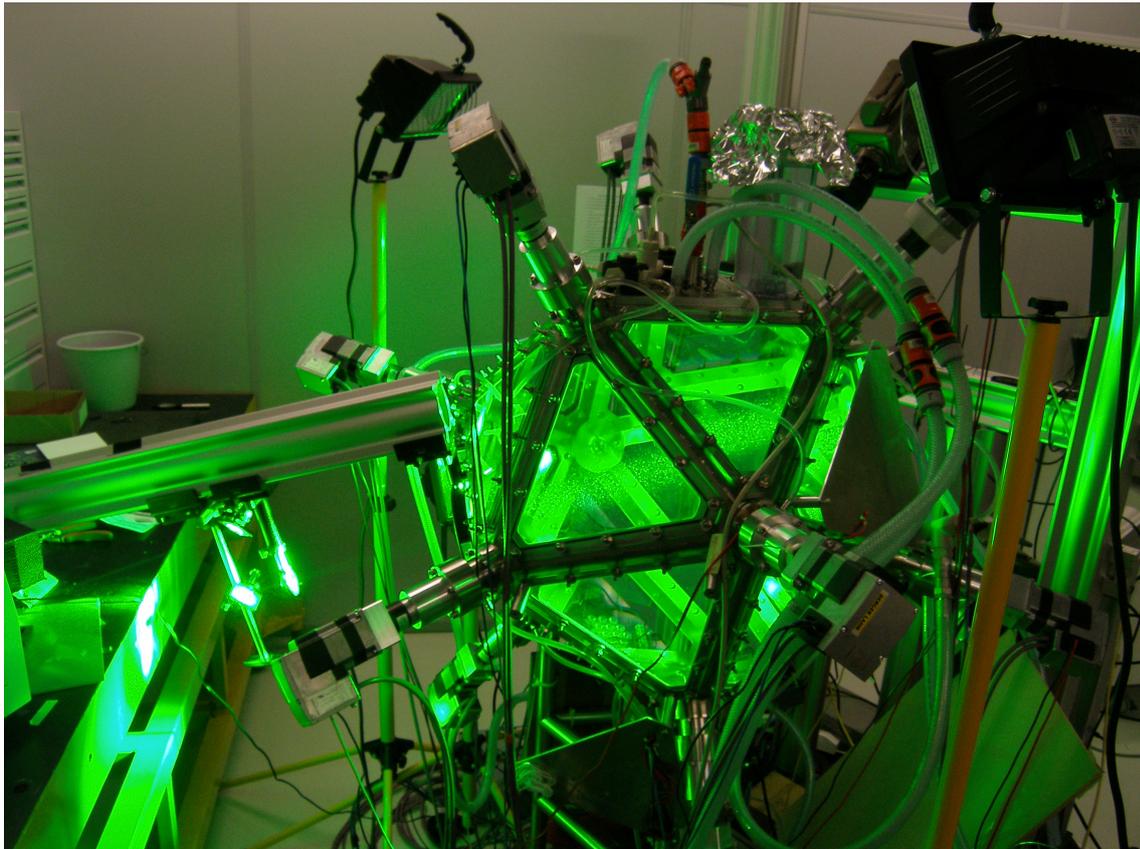


Figure V.12: The expanded laser beam illuminating the LEM; a 400 *mm* convex lens is sitting close to the window, the two optical components at the left are mirrors

lot more LEDs would be needed to register a significant improvement in the total light available.

### Particles

In order to use PTV, particles need to be added to the flow. The particles, however, should not disturb the flow or bias the measurement. To ensure this, the following properties have to be considered beforehand:

- the amount of light scattered by a particle
- the buoyancy of the particles ( = their density compared to the density of the surrounding fluid)
- the shape of the particle
- the response time to forces acting on the particle by the flow
- chemical properties
- prize and availability

The time scale,  $\tau_p$ , in which the particle reacts to changes in the flow, can be expressed in terms of the Stokes number:

$$St \equiv \frac{\tau_p}{\tau_\eta} \quad (\text{V.1})$$

where  $\tau_\eta$  is the smallest time scale of a turbulent flow (see section III.6).

In order to estimate  $\tau_p$ , one can picture a sphere falling through a fluid. From experiments it is known that such a sphere will also move some of the surrounding fluid. In other words, the mass of the sphere is increased. For low Reynolds numbers the particle is decelerated by the viscous Stokes drag force and accelerated by gravitation:

$$St = \frac{\rho_p + \frac{1}{2}\rho_f}{\rho_f} \frac{d^2}{18\nu} \frac{1}{\tau_\eta} = \frac{1}{18} \frac{\rho_p + \frac{1}{2}\rho_f}{\rho_f} \left(\frac{d}{\eta}\right)^2 \quad (\text{V.2})$$

where  $\rho_p$  and  $\rho_f$  are the densities of the particle and the fluid respectively, and  $d$  is the diameter of the particle.

For  $St \ll 1$ , the particles react quickly. According to equation (V.2) the ideal particle would have a density  $\rho_p < \rho_f$  and a size  $r < \eta$ . However, in our experiments it is better to match particle and fluid density to avoid buoyancy forces, and thus a preferred direction of motion for the particles<sup>2</sup>. Apart from that, the intensity of the scattered light depends on the radius squared. A compromise has to be found between fast reaction time and sufficiently high scattered intensities. In our case, the particles should be passive tracers of the flow.

Polystyrene and hollow glass spheres with densities of approximately  $1.06 \text{ g cm}^{-3}$  were chosen in our experiments. The particle size was  $42 \text{ }\mu\text{m}$  and  $90 \text{ }\mu\text{m}$  accordingly. The used Stokes numbers are shown in table VII.1.

## Calibration

For the calibration of the PTV system we exchange the top plate of the LEM (as shown in figure V.16) with the special calibration stage depicted in figure V.15. One or two linear stages can be mounted on the plexiglas plate. The whole system can be rotated, and the height of the mask can be adjusted. We built adaptor pieces to raise and fix the stage.

The masks employed were regular dot patterns with three bigger dots defining the center. They were created with Adobe Illustrator and printed on transparency foils. The foil in turn was either "glued" on to a glass plate<sup>3</sup> with either glycerol, or for smaller areas, high vacuum grease. The inclosure of small air bubbles must be avoided as they would cause additional reflections that distort the calibration images. Water entering the gap between glass surface and foil and thus changing the distance between the two must also be avoided.

Two masks with different dot spacings and sizes were used. One mask had a spacing between dots of  $10 \text{ mm}$  and  $3 \text{ mm}$  dot size, for the other mask the spacing

<sup>2</sup>It should be noted that the clustering of particles is related to the Stokes number and the densities  $\rho_p$  and  $\rho_f$ .

<sup>3</sup>trials with two-component glue or silicon failed

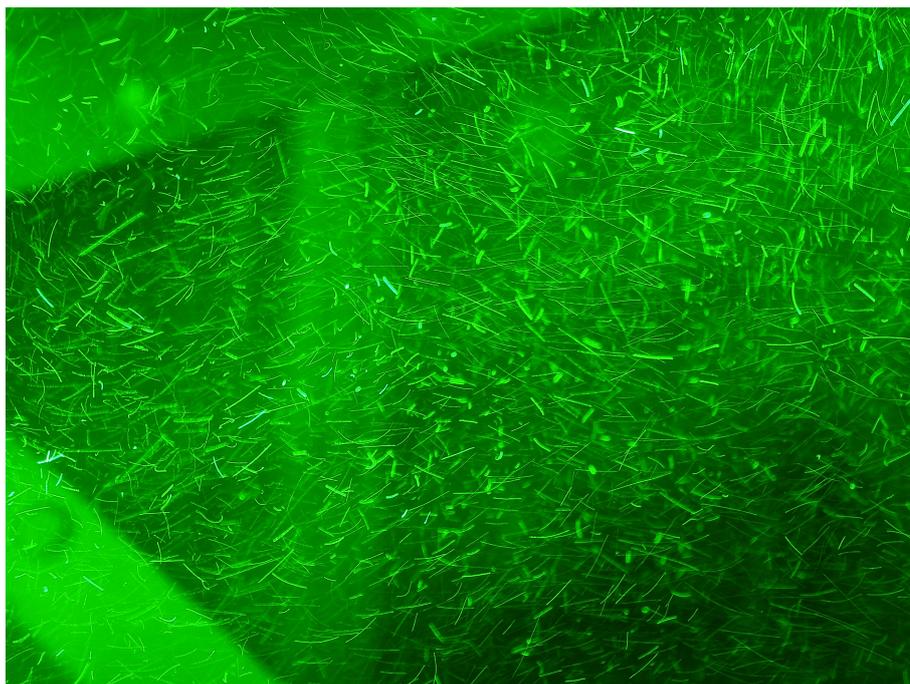


Figure V.13: A long exposure picture of  $90 \mu m$  particles in a medium turbulent flow

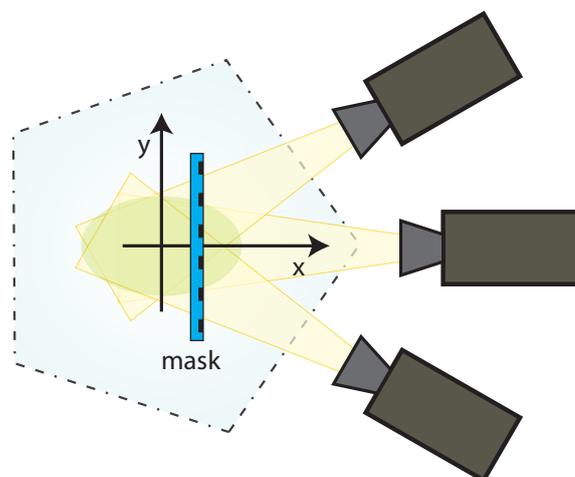


Figure V.14: Schematic top view of the camera arrangement: the three cameras are orientated at different angles inside the apparatus, such that they focus on the center of the tank.

was set to  $2.5 \text{ mm}$  and the dot size to  $0.9 \text{ mm}$ . Background illumination with a diffusive light source<sup>4</sup> resulted in the best image quality.

For calibration, a mask was moved along one axis within the tank, and short movies of approximately ten frames were recorded for each position of the mask. Figure V.14 provides a simple sketch of this setup. The axis along which the cali-

<sup>4</sup>For instance, a sheet of paper and a halogen bulb

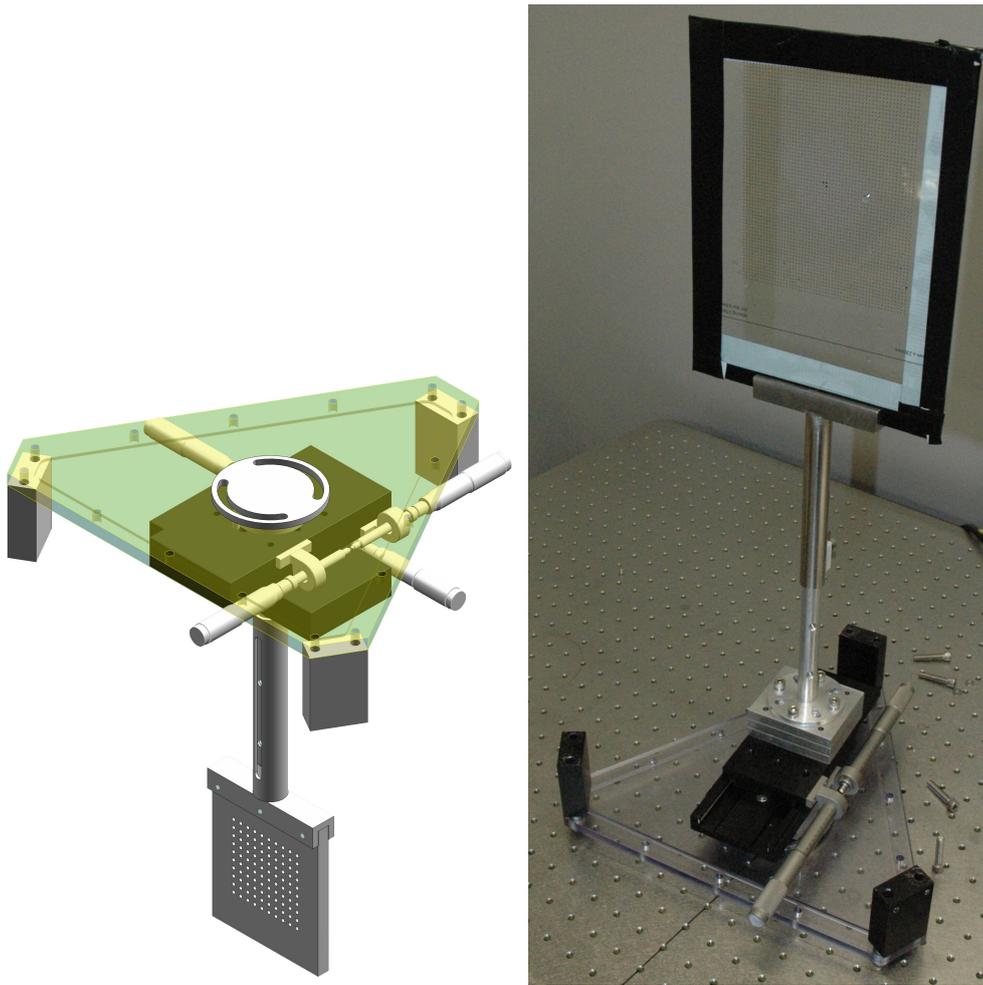


Figure V.15: figure on the left: CAD drawing of the calibration stage with two linear stages; figure on the right: assembled calibration stage before mounting in the LEM.

bration mask is moved, is defined as the  $x$ -axis in the experiments. We set  $x = 0$  as the center of the LEM.

A Matlab code written by Haitao Xu allows the identification of the dots in the calibration images. Using the three center dots, knowing the dot spacing of the mask, and the position of the mask in the  $x$ -direction, one can connect a set of 3D points with their projection on the CMOS chip of a camera. The parameters, which relate the picture on the image plane to the 3D lab system (see section IV.5.2), can be obtained using the method described in IV.5.3.

Relaxations of the holding structure would shift the cameras and thus the location of the image planes. For small perturbations it is sufficient to realign the image planes before the stereo matching step. A Matlab code (written by Haitao Xu) is used to obtain the average shift and rotation of each CMOS sensor from a PTV experiment. Before carrying out the real measurements the calibration was refined by storing shift and rotation in the configuration file. Hence a smaller position error was achieved.

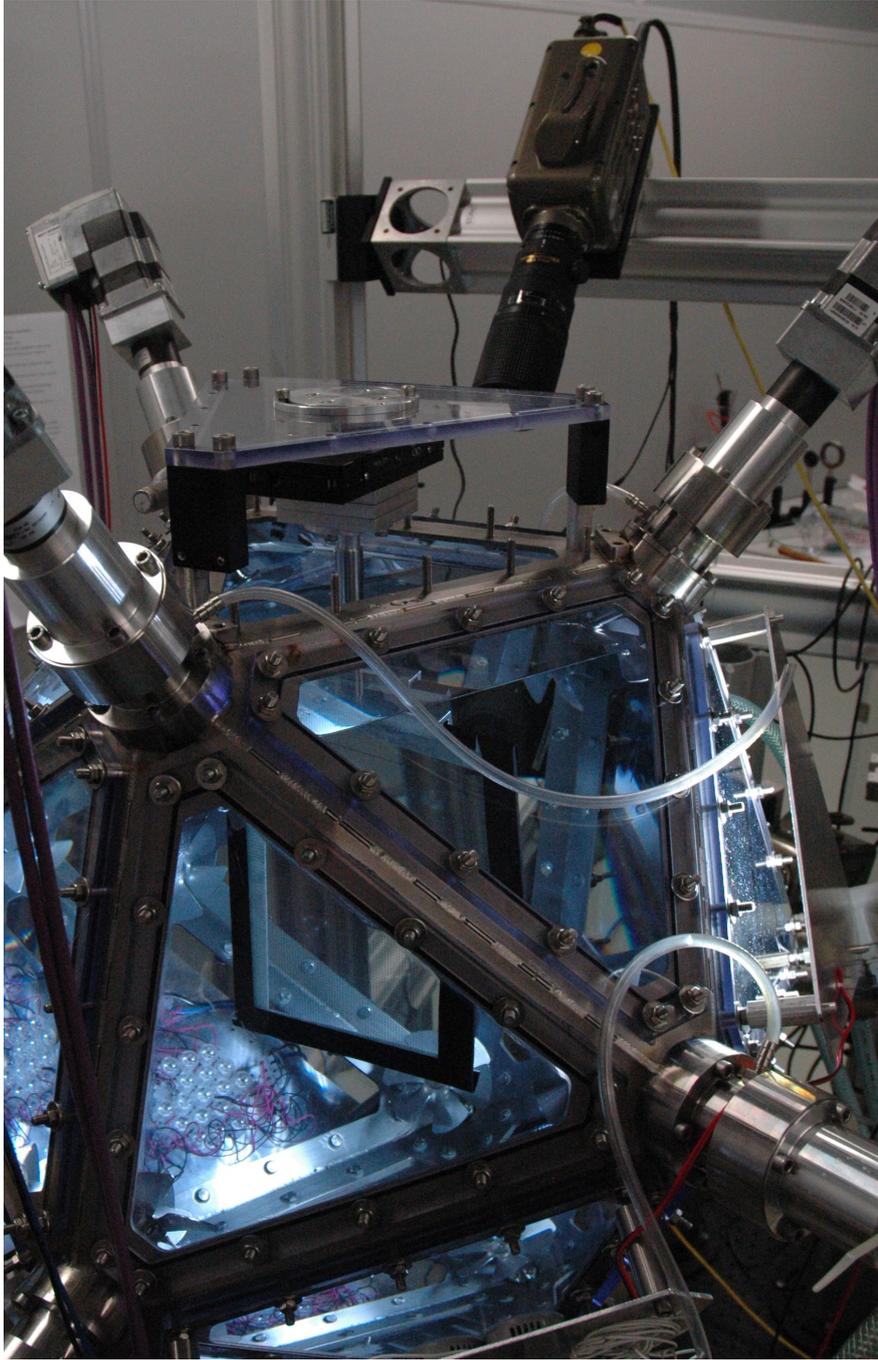


Figure V.16: The calibration stage mounted onto the LEM, note the three aluminum adaptor pieces needed to fixed the calibration stage, illumination of the mask can be provided by the LED plates or halogen bulbs

# VI Analysis of the Trajectories

## VI.1 Calculation of Velocity and Acceleration

Using the PTV technique described in section IV.5 and V.1.6 we get a series of particle trajectories<sup>1</sup>.

A trajectory can be represented as the position,  $\mathbf{x}(t)$ , of a particle at each frame  $t$ . The length of a trajectory can range from several ten to a few thousand frames. Noise and measurement errors  $\xi(t)$  bias the measured position. The measured position,  $\tilde{\mathbf{x}}(t)$ , is then

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \xi(t) \quad (\text{VI.1})$$

Both, the velocity,  $\mathbf{u}$ , and the acceleration,  $\mathbf{a}$ , are needed to calculate structure functions, correlation functions and other measures. To calculate the velocity  $\mathbf{v}(t)$ , the ansatz

$$\tilde{\mathbf{v}}(t) = \frac{\tilde{\mathbf{x}}(t+T) - \tilde{\mathbf{x}}(t)}{T} = \mathbf{v}(t) + \frac{\xi(t+T) - \xi(t)}{T} \quad (\text{VI.2})$$

is sensitive to noise. Consequently, the second derivative is even more sensitive to noise and measurement errors.

To overcome this issue, one can use a convolution (see Mordant *et al* [24]) of the original data,  $\tilde{\mathbf{x}}(t)$ , with some kernel function,  $k(\tau)$ , which smoothes the original data. Such a convolution is defined as:

$$k \otimes \tilde{\mathbf{x}} \Big|_t = \int_{-\infty}^{\infty} k(\tau) \cdot \tilde{\mathbf{x}}(\tau + t) d\tau \quad (\text{VI.3})$$

In the case of filtering  $k$  obeys  $k \otimes 1 = 1$  and  $k \geq 0$ . A useful rule for convolution is that the derivative of a convolution equals the derivative of its parts, i.e.

$$(k \otimes \tilde{\mathbf{x}})' = k' \otimes \tilde{\mathbf{x}} = k \otimes \tilde{\mathbf{x}}' \quad (\text{VI.4})$$

A Gaussian<sup>2</sup> kernel in the form  $k(\tau) = \frac{1}{\sqrt{\pi}\omega} \exp -\frac{\tau^2}{\omega^2}$  smoothes data effectively, and combined with rule VI.4 it becomes a powerful tool to evaluating derivatives of noisy data. The velocity is then:

$$k \otimes \tilde{\mathbf{v}} \Big|_t = k' \otimes \tilde{\mathbf{x}} \Big|_t = \int_{-\infty}^{\infty} k_v(\tau) \cdot \tilde{\mathbf{x}}(\tau + t) d\tau \quad (\text{VI.5})$$

with  $k_v = \frac{\partial k(\tau)}{\partial \tau} = -\frac{2\tau}{\sqrt{\pi}\omega^3} \exp\left(-\frac{\tau^2}{\omega^2}\right)$

<sup>1</sup>These are stored in several hundred to a few thousand trajectory files.

<sup>2</sup>It is  $\omega = \sqrt{2}\sigma$ .

and the acceleration can be computed using

$$k \otimes \tilde{\mathbf{a}} \Big|_t = k'' \otimes \tilde{\mathbf{x}} \Big|_t = \int_{-\infty}^{\infty} k_a(\tau) \cdot \tilde{\mathbf{x}}(\tau + t) d\tau$$

$$\text{with } k_a = \frac{\partial^2 k(\tau)}{\partial \tau^2} = \frac{2}{\sqrt{\pi}\omega^3} \left( \frac{2\tau^2}{\omega^2} - 1 \right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \quad (\text{VI.6})$$

Since  $k$  is known at the beginning, one can precalculate the new kernels  $k_v$  and  $k_a$ . Furthermore, we can limit the integral to the range  $[-L; L]$  since the contribution of values decays with  $\exp(-\tau^2)$ . Because we have discrete values and not a continuous function, the integral is evaluated as a sum. To calculate the velocity and acceleration at time  $t$ , for example, we need to evaluate  $\sum_{-L}^{+L} k_v \cdot \tilde{\mathbf{x}}(\tau + t)$  and  $\sum_{-L}^{+L} k_a \cdot \tilde{\mathbf{x}}(\tau + t)$ . This operation is constant in time and can be done easily by a computer. But, since we are truncating the integral, we have to normalize the kernel functions. The discrete functions are

$$k_v(\tau) = A_v \tau \exp\left(-\frac{\tau^2}{\omega^2}\right) + B_v \quad \text{and} \quad k_a(\tau) = A_a \exp\left(-\frac{\tau^2}{\omega^2}\right) \left[ \frac{2\tau^2}{\omega^2} - 1 \right] + B_a$$

The parameters  $A_v$ ,  $B_v$ ,  $A_a$  and  $B_a$  can be determined using the known derivatives

$$k_v \otimes 1 = 0 \quad k_v \otimes \tau = 1 \quad (\text{VI.7})$$

$$k_a \otimes 1 = 0 \quad k_a \otimes \tau^2 = 2 \quad (\text{VI.8})$$

### VI.1.1 Calculating $A_v$ and $B_v$

$$1 = \sum_{-L}^L k_v(\tau) \cdot (\tau) = \sum_{-L}^L \left( A_v \tau \exp\left(-\frac{\tau^2}{\omega^2}\right) + B_v \right) \cdot \tau$$

$$= A_v \sum_{-L}^L \tau^2 \exp\left(-\frac{\tau^2}{\omega^2}\right) \quad (\text{VI.9})$$

$$\Leftrightarrow A_v = \frac{1}{\sum_{-L}^L \tau^2 \exp\left(-\frac{\tau^2}{\omega^2}\right)}$$

$$0 = \sum_{-L}^L k_v(\tau) = \sum_{-L}^L \left( A_v \tau \exp\left(-\frac{\tau^2}{\omega^2}\right) + B_v \right)$$

$$= (2L + 1)B_v + A_v \sum_{-L}^L \tau \exp\left(-\frac{\tau^2}{\omega^2}\right) \quad (\text{VI.10})$$

$$= (2L + 1)B_v + 0$$

$$\Leftrightarrow B_v = 0$$

### VI.1.2 Calculating $A_a$ and $B_a$

$$\begin{aligned}
2 &= \sum_{-L}^L k_a(\tau) \cdot (\tau^2) = \sum_{-L}^L \left[ A_a \exp\left(-\frac{\tau^2}{\omega^2}\right) \left(\frac{2\tau^2}{\omega^2} - 1\right) + B_a \right] \cdot \tau^2 \\
&= A_a \sum_{-L}^L \left[ \tau^2 \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right] + B_a \sum_{-L}^L \tau^2 \\
&= A_a \sum_{-L}^L \left[ \tau^2 \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right] + B_a \frac{L(L+1)(2L+1)}{3}
\end{aligned} \tag{VI.11}$$

$$\begin{aligned}
0 &= \sum_{-L}^L k_a(\tau) = \sum_{-L}^L \left( A_a \exp\left(-\frac{\tau^2}{\omega^2}\right) \left[\frac{2\tau^2}{\omega^2} - 1\right] + B_a \right) \\
\Leftrightarrow A_a &= \frac{-(2L+1)B_a}{\sum_{-L}^L \left[ \exp\left(-\frac{\tau^2}{\omega^2}\right) \left(\frac{2\tau^2}{\omega^2} - 1\right) \right]}
\end{aligned} \tag{VI.12}$$

Combining equations (VI.11) and (VI.12) one obtains

$$\begin{aligned}
2 &= B_a \left( \frac{-(2L+1) \sum_{-L}^L \left( \tau^2 \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right)}{\sum_{-L}^L \left( \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right)} + \frac{L(L+1)(2L+1)}{3} \right) \\
\Leftrightarrow B_a &= 2 \left( \frac{-(2L+1) \sum_{-L}^L \left( \tau^2 \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right)}{\sum_{-L}^L \left( \left(\frac{2\tau^2}{\omega^2} - 1\right) \exp\left(-\frac{\tau^2}{\omega^2}\right) \right)} + \frac{L(L+1)(2L+1)}{3} \right)^{-1}
\end{aligned} \tag{VI.13}$$

To find the right values for  $A_v, A_a$  and  $B_a$  one has to calculate the sums with given values  $L$  and  $\omega$ . Velocity and acceleration are calculated with different filter width. The convolution with vectors, e.g.  $k_v \otimes \tilde{\mathbf{x}} \Big|_t$ , can be calculated by employing to convolution to each component of the vector. To reduce the uncertainty in the position, we also smooth the measured position,  $\tilde{\mathbf{x}}$ , with the same  $L$  and  $\omega$  used also for the velocity.

The analysis of the trajectories is implemented in C++ using object orientated-programming and standard libraries like `std::vector` to store trajectories in memory.

### VI.1.3 Interpolated points

Due to light intensity fluctuations, particles blocking the line of sight, hardware and software problems the trajectory can break up into smaller pieces. The PTV code is able to extrapolate small gaps up to 3 frames ahead and continue if a real point is close to the extrapolation. The Track Connection method (see section IV.5.7) instead is able to connect tracks with gaps of several tenth or hundreds of missing

frames. Since such a long interpolation might bias the evaluation, one has to check the contribution of interpolated points to the statistics. One attempt is to discard every velocity and acceleration value with interpolated points more than a certain threshold. Although this is computationally cheap the bias introduced by interpolated points to the calculation depends on the shape of the kernel. Thus calculating the contribution to the kernel would allow to use more points. Let

$$I(t) = \begin{cases} 1 & \text{if interpolated} \\ 0 & \text{otherwise} \end{cases}$$

and  $i_v = C_v \cdot \|k_v\|$  as well as  $i_a = C_a \cdot \|k_a\|$  the test kernel functions. Since they are kernel functions they have to obey  $i_v \otimes 1 = i_a \otimes 1 = 1$ . The influence of an extrapolation can be limited by discarding calculations with

$$i_v \otimes I \geq I_{vel,max} \quad \text{or} \quad i_a \otimes I \geq I_{acc,max}$$

For  $I(t) = const = 1$  the convolution returns 1, for a trajectory containing no extrapolated points the contribution is 0. Usually experiments are performed with many frames per  $\tau_\eta$  hence the extrapolation for small gaps won't introduce big errors and the extrapolated points can be used.

The analysis code written by Haitao Xu in our group discards a computation if more than 3 points at a filter window of 27 frames are interpolated. This corresponds in the case of 3 consequent interpolated points to  $I_{acc,max} \approx 0.3$  and  $I_{vel,max} \approx 0.45$ . To feel certain

$$I_{acc,max} = 0.20 \quad \text{and} \quad I_{vel,max} = 0.35$$

will be used.

The test has to be done only for one component and the acceleration can be tested before the velocity to avoid redundant calculations.

## VI.2 Probability density functions

The probability  $P$  of a statistical property  $x$  to be within  $a \leq x \leq b$  is:

$$P(a \leq x \leq b) = \int_a^b p(x) dx \quad \text{with} \quad \int_{-\infty}^{\infty} p(x) dx = 1 \quad \text{and} \quad p \geq 0 \quad (\text{VI.14})$$

where  $p(x)$  is the probability density function (pDF). The  $k$ -th moment of a probability function is defined as:

$$M_k \equiv \int_{-\infty}^{\infty} p(x) x^k dx \quad (\text{VI.15})$$

and the  $k$ -th central moment as:

$$C_k \equiv \int_{-\infty}^{\infty} p(x) (x - \mu)^k dx \quad (\text{VI.16})$$

where  $\mu$  is the mean value of the pdf and the first moment of  $p$ . The second central moment of the PDF is the variance of the distribution. The PDFs of velocity and acceleration can be easily created using binning. For the first file being processed the minimum and maximum value of velocity and acceleration components and particle positions are found. These intervals are divided into  $N_{bins}$  bins, which are basically counters. Extra bins are added on both sides so that the statistics are not biased by the rather random choice of the minimum and maximum. For each velocity and acceleration value computed the appropriate bin is calculated and the counter is incremented by 1. Then the probability of measuring a value corresponding to the  $i$ -th bin  $X_i$  is:

$$P(X_i) = \frac{n_i}{n_{tot}} \quad (\text{VI.17})$$

where  $n_i$  denotes the number of counts in this bin and  $n_{tot} = \sum_i n_i$  the total number of counts. Using equations (VI.14) and (VI.17) it is:

$$\begin{aligned} P(X_i) &= \frac{n_i}{n_{tot}} = \int_{\min(X_i)}^{\max(X_i)} p(x) dx \\ &\approx p(x) \Delta x \\ \Leftrightarrow p(x) &= \frac{n_i}{n_{tot}} \frac{1}{\Delta x} \end{aligned} \quad (\text{VI.18})$$

where  $x = \frac{\max(X_i) + \min(X_i)}{2}$  and  $\Delta x = \max(X_i) - \min(X_i)$ .

To compare different PDFs it is useful to normalize them using the statistical z-transform. This is done in 2 steps. First the mean  $\mu$  and standard deviation  $\sigma$  are calculated using equations (VI.15) and (VI.16). Then the probability density is renormalized as a function of  $\frac{x-\mu}{\sigma}$ . Thus the renormalised PDF is:

$$p\left(\frac{x-\mu}{\sigma}\right) = \frac{n_i}{n_{tot}} \frac{\Delta x}{\sigma} = \frac{n_i \sigma}{n_{tot} \Delta x} \quad (\text{VI.19})$$

## VI.3 Subtracting the mean flow field

To obtain the fluctuating velocity one has to subtract the mean flow field. Therefore the measurement volume is subdivided into  $n \times n \times n$  spatial bins. Particles passing through such a spatial bin will allow to calculate the mean and rms values of acceleration and velocity. After processing all movies the mean flow field can be approximated as an interpolation of the obtained mean velocities.

To create a steady mean flow field a trilinear interpolation<sup>3</sup> instead of a nearest neighbor method was chosen as an interpolation method.

It is assumed that only direct neighbors are important. Let  $\mathbf{C} = (x, y, z)$  be the point whose value has to be interpolated and  $\mathbf{C}_{XYZ}$  the eight vortices of a cuboid around  $\mathbf{C}$ . The cuboid is rescaled to a cube of edge length one<sup>4</sup>.  $X$ ,  $Y$  and  $Z$  denote

<sup>3</sup>A trilinear interpolation is a linear interpolation in 3D.

<sup>4</sup>Therefore it is  $x, y, z \in [0, 1]$

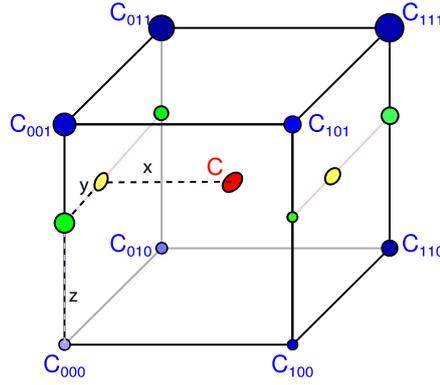


Figure VI.1: Trilinear interpolation for a value at point  $\mathbf{C}$ . The value at a point is visualised by the size and shape of the circle, green corresponds to the first interpolation  $i$ , yellow to the second step  $j$ . The cell is a cube with an edge length of 1.

the coordinates of a corner. Furthermore  $v(\mathbf{C})$  denotes the value at position  $\mathbf{C}$  and  $v_{XYZ}$  the value at a corner  $\mathbf{C}_{XYZ}$ .

If the number of statistics in bin is not sufficient, the mean over the all usable spatial bins will be used.

The interpolation is done by following three linear interpolation steps.

1. Interpolate the values along the  $z$ -axis. The four interpolated values are:

$$\begin{aligned}
 i_1 &= v_{000} + z \cdot (v_{001} - v_{000}) \\
 i_2 &= v_{100} + z \cdot (v_{101} - v_{100}) \\
 i_3 &= v_{010} + z \cdot (v_{011} - v_{010}) \\
 i_4 &= v_{110} + z \cdot (v_{111} - v_{110})
 \end{aligned} \tag{VI.20}$$

In figure VI.1 this is visualized by the four green dots. The size of the green dots is a size between their upper and bottom neighbor.

2. Interpolating between the interpolated points in  $y$  direction

$$\begin{aligned}
 j_1 &= i_1 + y (i_4 - i_1) \\
 j_2 &= i_2 + y (i_3 - i_2)
 \end{aligned} \tag{VI.21}$$

This corresponds to the two yellow circles in figure VI.1.

3. by interpolating in  $x$ -direction we find

$$\begin{aligned}
 v(\mathbf{C}) &= j_1 + x (j_2 - j_1) \\
 &= i_1 + y (i_4 - i_1) + x i_2 + x y (i_3 - i_2) - x i_1 - x y (i_4 - i_1) \\
 &= i_1 (1 - x - y + x y) + i_2 (x - x y) + i_3 (x y) + i_4 (y - x y)
 \end{aligned} \tag{VI.22}$$

Combining all equations we obtain

$$\begin{aligned}
v(\mathbf{C}) &= v_{000} + z(v_{001} - v_{000}) \\
&+ y v_{110} + y z(v_{111} - v_{110}) \\
&- y v_{000} - y z(v_{001} - v_{000}) \\
&+ x v_{100} + x z(v_{101} - v_{100}) \\
&+ x y v_{010} + x y z(v_{011} - v_{010}) \\
&- x y v_{100} - x y z(v_{101} - v_{100}) \\
&- x v_{000} - x z(v_{001} - v_{000}) \\
&- x y v_{110} - x y z(v_{111} - v_{110}) \\
&+ x y v_{000} + y z(v_{001} - v_{000})
\end{aligned} \tag{VI.23}$$

Rearranging (VI.23)

$$\begin{aligned}
v(\mathbf{C}) &= v_{000} (1 - x)(1 - y)(1 - z) \\
&+ v_{001} (1 - x)(1 - y) z \\
&+ v_{010} x y (1 - z) \\
&+ v_{011} x y z \\
&+ v_{100} x (1 - y) (1 - z) \\
&+ v_{101} x (1 - y) z \\
&+ v_{110} y (1 - x) (1 - z) \\
&+ v_{111} y (1 - x) z
\end{aligned} \tag{VI.24}$$

Before calculating Eulerian and Lagrangian statistics the mean velocity field and the mean acceleration field were subtracted.

## VI.4 Linear approximation of the flow field

The mean flow field  $\mathbf{U}(\mathbf{r})$  can be approximated as:

$$\mathbf{u} \approx \mathbf{u}(\mathbf{r}_0) + \underline{\underline{\mathbf{M}}} (\mathbf{r} - \mathbf{r}_0) \tag{VI.25}$$

where  $\underline{\underline{\mathbf{M}}}$  is an approximation of the Jacobian  $\nabla \mathbf{u}(\mathbf{r}_0)$  at the point  $\mathbf{r}_0$ .

$\underline{\underline{\mathbf{M}}}$  can be obtained from the mean flow using the following procedure.  $N$  velocity vectors  $\mathbf{u}_i = (u_i^x \mid u_i^y \mid u_i^z)$  at their corresponding position  $\mathbf{r}_i \equiv (x_i \mid y_i \mid z_i)$  have been measured. Lets define  $\delta_i = (\delta_i^x \mid \delta_i^y \mid \delta_i^z) = (x_i - r_0^x \mid y_i - r_0^y \mid z_i - r_0^z)$  and  $\mathbf{u}(\mathbf{r}_0) = (u_0^x \mid u_0^y \mid u_0^z)$ . Applying an arbitrary pair  $\mathbf{u}_i, \delta_i$  to equation (VI.25) yields the following three equations

$$u_i^x = u_0^x + M_{11} \delta_i^x + M_{12} \delta_i^y + M_{13} \delta_i^z \tag{VI.26a}$$

$$u_i^y = u_0^y + M_{21} \delta_i^x + M_{22} \delta_i^y + M_{23} \delta_i^z \tag{VI.26b}$$

$$u_i^z = u_0^z + M_{31} \delta_i^x + M_{32} \delta_i^y + M_{33} \delta_i^z \tag{VI.26c}$$

Nevertheless, the continuity equation dictates:

$$M_{11} + M_{22} + M_{33} = 0 \quad (\text{VI.27})$$

Rearranging equation (VI.25) and employing the constraint (VI.27) gives three independent overdetermined systems of linear equations:

$$\begin{pmatrix} \delta_1^x & \delta_1^y & \delta_1^z & 1 \\ \delta_2^x & \delta_2^y & \delta_2^z & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \delta_N^x & \delta_N^y & \delta_N^z & 1 \end{pmatrix} \cdot \begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \\ v_0^x \end{pmatrix} = \begin{pmatrix} v_1^x \\ v_2^x \\ \vdots \\ v_N^x \end{pmatrix} \quad (\text{VI.28a})$$

$$\begin{pmatrix} \delta_1^y & \delta_1^y & \delta_1^z & 1 \\ \delta_2^y & \delta_2^y & \delta_2^z & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \delta_N^y & \delta_N^y & \delta_N^z & 1 \end{pmatrix} \cdot \begin{pmatrix} M_{21} \\ M_{22} \\ M_{23} \\ v_0^y \end{pmatrix} = \begin{pmatrix} v_1^y \\ v_2^y \\ \vdots \\ v_N^y \end{pmatrix} \quad (\text{VI.28b})$$

$$\begin{pmatrix} \delta_1^x & \delta_1^y & 1 \\ \delta_2^x & \delta_2^y & 1 \\ \vdots & \vdots & \vdots \\ \delta_N^x & \delta_N^y & 1 \end{pmatrix} \cdot \begin{pmatrix} M_{31} \\ M_{32} \\ v_0^z \end{pmatrix} = \begin{pmatrix} v_1^z + (M_{11} + M_{22})\delta_2^z \\ v_2^z + (M_{11} + M_{22})\delta_2^z \\ \vdots \\ v_N^z + (M_{11} + M_{22})\delta_N^z \end{pmatrix} \quad (\text{VI.28c})$$

where the  $M_{ij}$  and  $u_0^x, u_0^y, u_0^z$  are unknowns.

For an overdetermined systems of linear equations  $\underline{\underline{\mathbf{A}}}\mathbf{x} = \mathbf{b}$ , with  $\underline{\underline{\mathbf{A}}}$  a  $n \times m$ ,  $\mathbf{x}$  a  $m \times 1$  and  $\mathbf{b}$  a  $n \times 1$  matrix/vector, the inverse of  $\underline{\underline{\mathbf{A}}}$  does not exist if  $n \neq m$ . However, one solution of the system is

$$\mathbf{x} = \underline{\underline{\mathbf{A}}}^+ \mathbf{b} \quad (\text{VI.29})$$

where  $\underline{\underline{\mathbf{A}}}^+$  is the pseudo-inverse of  $\underline{\underline{\mathbf{A}}}$ , which can be calculated from a singular value decomposition of  $\underline{\underline{\mathbf{A}}}$ . A second solution is the use of an optimization scheme which finds

$$\min (\|Ax - b\|) \quad (\text{VI.30})$$

We are using equation (VI.29) to determine the matrix  $\underline{\underline{\mathbf{M}}}$ .

## VI.5 Calculation of the Eulerian autocorrelation $R_{ij}$ and second-order-structure function $D_{ij}$

To calculate the Eulerian structure functions and autocorrelation we have to convert the Lagrangian information to Eulerian information. The PTV system allows to measure to follow on the order of 100 particles at a time in the measurement volume. Thus the mean flow field is too coarse to compute Eulerian properties from the mean flow field. Assuming a locally isotropic flow with no mean flow one can obtain Eulerian structure functions and autocorrelation by the following procedure:

1. create a list of all points at a frame<sup>5</sup>  $f$
2. pick two vectors (most likely velocity or acceleration) from this list:  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . These vectors are separated by  $r \mathbf{e}_r$  where  $\mathbf{e}_r$  denotes the unit vector of  $\mathbf{r}$ .
3. As explained in section III.6,  $D_{ij}(\mathbf{r})$  and  $R_{ij}(\mathbf{r})$  depend only on the length of  $\mathbf{r}$  in a locally isotropic flow with subtracted mean flow. Thus one can calculate  $D_{LL}(\|\mathbf{r}\|) = \langle [\mathbf{p}_1 \cdot \mathbf{e}_r - \mathbf{p}_2 \cdot \mathbf{e}_r]^2 \rangle$  and  $R_{LL}(\|\mathbf{r}\|) = \langle [\mathbf{p}'_1 \cdot \mathbf{e}_r] [\mathbf{p}'_2 \cdot \mathbf{e}_r] \rangle$  with  $\mathbf{p}' = \mathbf{p} - \langle \mathbf{p} \rangle$  and average the result employing a binning technique for  $r$ .
4. Analogous to the prior step one can gain the transverse structure function and auto correlation by substituting  $\mathbf{e}_r$  for a vector  $\mathbf{e}_n$  perpendicular to  $\mathbf{e}_r$ .
5. The ensemble average can be computed by using all possible combinations of points at a time step and doing this for each frame.

A transverse vector  $\mathbf{e}_n = \mathbf{e}_a \times \mathbf{e}_r$  with an arbitrarily set vector  $\mathbf{e}_a$  can be used to obtain the transverse second order structure function and autocorrelation. A second transverse vector  $\mathbf{e}_k$  can be obtained from  $\mathbf{e}_k = \mathbf{e}_n \times \mathbf{e}_r$ . However, the cross product vanishes for  $\mathbf{e}_r = \mathbf{e}_a$ . Moreover, in the case both when vectors are close to parallel the measurement error in  $\mathbf{e}_r$  results in a bigger uncertainty in  $\mathbf{e}_n$ .

Nevertheless we set  $\mathbf{e}_a$  to be parallel to the z- axis  $\mathbf{e}_z$  to calculate the two normal components. If  $\mathbf{e}_r$  and  $\mathbf{e}_z$  are close to parallel<sup>6</sup> they were set  $\mathbf{e}_n = \mathbf{e}_x$  and  $\mathbf{e}_k = \mathbf{e}_y$ .

In order to check a angular dependence of  $D_{ij}$  and  $R_{ij}$  one can bin the both measures according to the spherical coordinates<sup>7</sup> of the unit vector  $\mathbf{e}_r(r = 1, \phi, \theta)$ . Since the sign of  $\mathbf{r}$  is not important, we can map  $\mathbf{r}$  on a half sphere. Changing the sign of  $\mathbf{r}$  can be done by changing the sign of  $\theta$  and setting  $\phi$  to  $\phi + 180^\circ$ .

In order to fill the bins approximately evenly, each bin  $A_{\theta, \phi}$  has to have the same area. This is achieved by a constant spacing in  $\phi$  but a adjusted spacing in  $\theta$ . The area of the  $i$ -th slice (limited by  $\theta_i$  and  $\theta_{i-1}$ ) of an unit- sphere is

$$A_i = 2\pi r(h_i - h_{i-1}) = 2\pi (\cos(\theta_i) - \cos(\theta_{i-1})) \quad (\text{VI.31})$$

Each slice has to cover

$$A_i = \frac{A_{total}}{m} = \frac{2\pi}{m}$$

Combining both we get  $\theta_i = \arccos\left(\frac{i}{m}\right)$ . The number  $i$  of the bin corresponding to an (arbitrary) angle  $\theta$  is the first  $i$  with  $\theta_{i+1} > \theta$ .

<sup>5</sup>time step and frame are considered to be equal

<sup>6</sup>they are approximately parallel if  $\|z\| > 100 \cdot (\|x\| + \|y\|)$

<sup>7</sup>It is  $\theta = \arccos(\mathbf{e}_r \cdot \mathbf{e}_z)$  and  $\phi = \arccos\left(\frac{\|\mathbf{e}_r \cdot \mathbf{e}_x\|}{\sqrt{(\mathbf{e}_r \cdot \mathbf{e}_x)^2 + (\mathbf{e}_r \cdot \mathbf{e}_y)^2}}\right)$



## VII Results

### VII.1 Experiments performed

Two arrangements of the PTV system have been used to investigate flows with different strengths and types of forcing, in each case using the stainless steel propellers. An overview of the results as well as of the corresponding Stokes numbers is provided in table VII.1 and table VII.2.

#### VII.1.1 Big measurement volume experiments

Using 60mm Nikon objectives, a relatively big measurement volume of approximately  $15 \times 10 \times 10 [cm^3]$  situated in the center of the LEM could be resolved.  $90\mu m$  “LiteSphere” particles manufactured by MO-SCI cooperation were used as tracers. The 3D spatial resolution was  $0.3 \frac{mm}{pixel}$ .

**Clockwise and Counterclockwise Forcing** Experiments with all motors at the same speed were done at  $+300 rpm$ ,  $+500 rpm$ ,  $+675 rpm$ ,  $+1500 rpm$ ,  $+2250 rpm$ ,  $+2800 rpm$ ,  $+5000 rpm$  and  $-2800 rpm$ . The difference in the direction of the rotation will be denoted by clockwise (CW) for positive frequencies and counterclockwise (CCW) in the other case. The motors of the LEM are able to run at up to  $+5000 rpm$ . However, due to illumination problems it was not possible to obtain enough data at this speed. Therefore, this experimental run is excluded.

**Anisotropic Forcing** The six motors at the top and at the bottom were pushing with  $+1000 rpm$ , the six propellers at the ‘equator’ were not rotating.

**Random Forcing** One experiment was performed using randomly picked frequencies  $f_i$  for each motor. Every 30 seconds new frequencies were chosen. From previous experiments in a French-Washing machine we know that  $u' \propto f$ . Inserting this into equation<sup>1</sup> (III.22) yields  $\varepsilon \propto f^3$ . Therefore, we enforced  $\frac{1}{12} \sum_{i=1}^{12} f_i^3 = f^3 = (500 rpm)^3$  and  $f_i \in [300 rpm; 700 rpm]$  to ensure a constant energy injection rate,  $\varepsilon$ .

#### VII.1.2 Small measurement volume experiments

Using 200 mm Nikon objectives, a smaller measurement volume of approximately  $15 \times 10 \times 10 [mm^3]$  situated in the center of the LEM could be resolved.  $42 \mu m$

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<sup>1</sup> $\varepsilon \sim \frac{U^3}{L}$

polystyrene particles (Duke Scientific) were used as tracers. The three dimensional spatial resolution was  $30 \frac{\mu m}{pixel}$ .

**Clockwise Forcing** Experiments with all motors at the same speed were done at  $+300 \text{ rpm}$ ,  $+500 \text{ rpm}$ ,  $+675 \text{ rpm}$ .

**Random Forcing** The random forcing described in VII.1.1 was repeated under the same forcing conditions.

In the case of anisotropic and random forcing, the frequencies of the propellers were not uniform, and thus no clear definition of the motor frequency can be given. We estimate the characteristic frequency of random and anisotropic forcing as  $+500 \text{ rpm}$  and  $+1000 \text{ rpm}$  respectively.

## VII.2 Selection of the Filter Width for the differentiating kernel and the parameters for the track connection code

To use the track connection method [23] described in section IV.5.7, the spatial and temporal search distance as well as the measurement error have to be specified. The measurement error  $\sigma_p$  is known from the calibration, and the spatial search distance,  $d_s$ , is limited by the average distance between particles. Therefore, we measured the mean,  $\langle N_p \rangle$ , and the standard deviation  $\langle (\langle N_p \rangle - N_p)^2 \rangle^{1/2}$  of particles per frame,  $N_p$ , in the biggest trajectory file. Since the size,  $V$ , of the observation volume is known, the spatial search distance  $d_s$  was set to

$$d_s \leq \frac{\sqrt[3]{V}}{\langle N_p \rangle + \langle (\langle N_p \rangle - N_p)^2 \rangle^{1/2}} \quad (\text{VII.1})$$

The term  $\langle N_p \rangle + \langle (\langle N_p \rangle - N_p)^2 \rangle^{1/2}$  estimates the maximum number of particles in the volume, whereas  $\sqrt[3]{V}$  returns the characteristic length of the observation volume. Equation (VII.1) can be seen as an estimate of the minimum average distance between particles. The temporal search distance,  $t_s$ , was varied to determine its optimal value which allows the code to find the maximal number of connections with a negligible number of wrong connections.  $t_s$  was set to 150 frames for all experiments.

As described in section VI.1, the derivatives of a gaussian kernel are used to calculate velocity and acceleration. The filter width  $W_f = 2L + 1$  and the filter variance  $\omega_f$  have to be chosen such that the statistics obtained do not depend on the choice of  $W_f$  and  $\omega_f$ . Also, they should be as small as possible to use the maximum amount of data and to see short time events. Each experiment has been processed with different filter widths and a ratio of  $W_f$  to  $\omega_f$  of approximately 3 to determine the optimal filter widths. The ratio was chosen according to previous experiences of our group with the PTV system (see e.g. [26]).

The measurement errors in position strongly influence the computation of properties based on velocity and acceleration. It can be shown[26] that applying the acceleration convolution filter on pure delta-correlated (white) noise produces an acceleration variance which is

$$\langle a^2 \rangle \propto W_f^{-5} \quad (\text{VII.2})$$

For longer filter windows the noise contribution gets smaller until the curve shows a kink at a certain  $W_f^{acc}$ . At this point the contribution of the filter width is such that the contribution of noise is small, and short time events of acceleration can still be measured. Therefore, the r.m.s. of acceleration and the theoretical  $-2.5$  power-law are log-log plotted to determine the optimal filter width  $W_f^{acc}$ . Because showing all 14 runs would go beyond the aim of this text, the run at  $+2800 \text{ rpm}$  (figures VII.1 and VII.2) has been picked as a representative example.

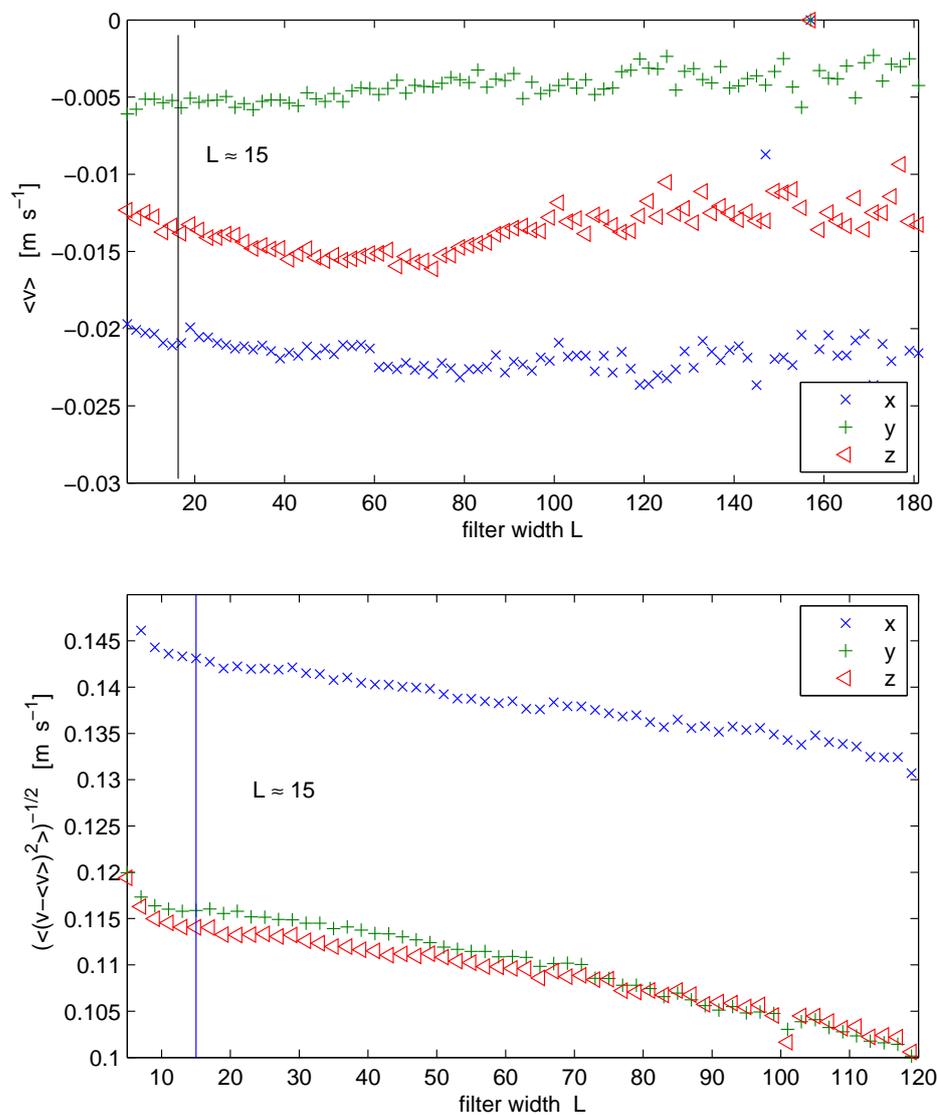


Figure VII.1: Velocity dependence on the filter width for the +2800 rpm run. Velocity statistics are less sensitive to the filter width than acceleration statistics (see figure VII.2).

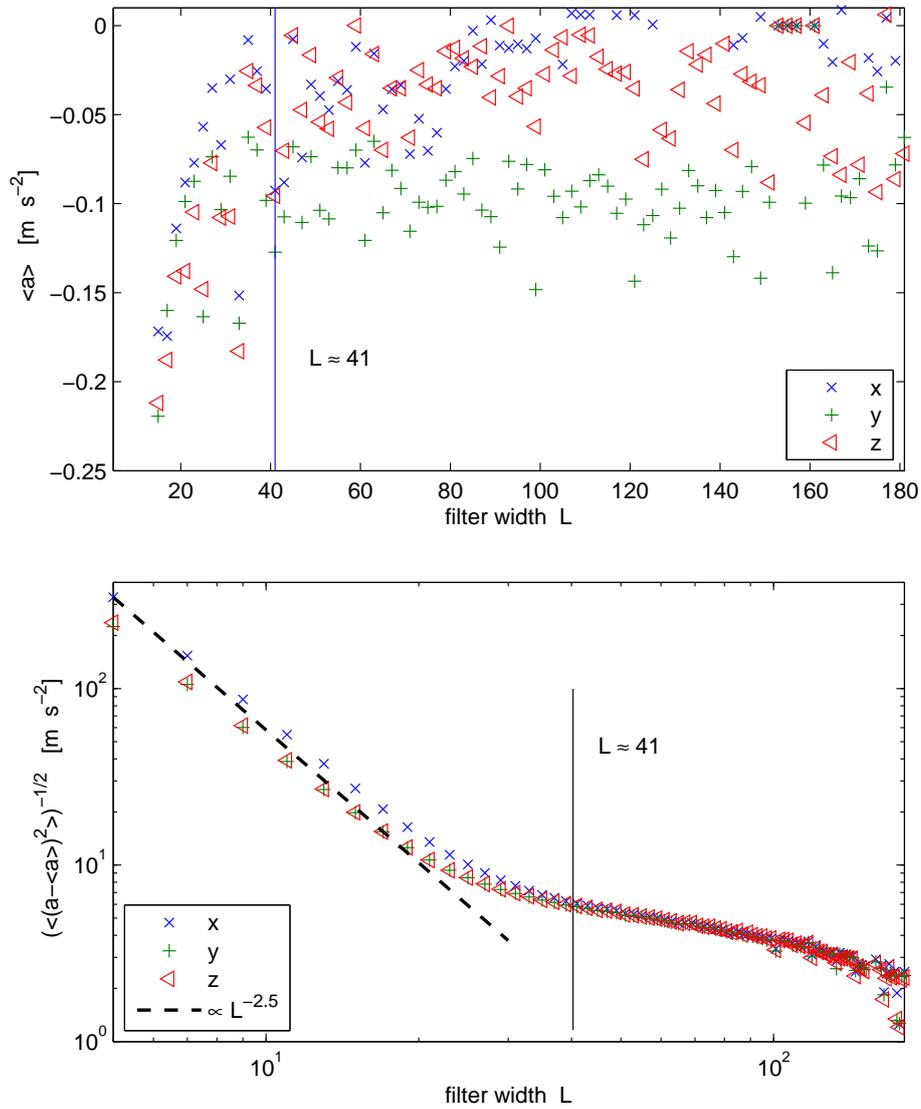


Figure VII.2: Acceleration dependence on the filter width for the +2800 rpm run. Note the kink in the acceleration rms at a filter width of approximately 41 frames;

### VII.3 Basic properties of the apparatus

To find the mean and the fluctuating velocity the moments, of the probability density function (PDF) of velocity were used. The fluctuating velocity,  $u'$ , was estimated from the standard deviation (r.m.s.) of the velocity probability density function.

The dependence of the fluctuating and mean velocity on the propeller frequency,  $f$ , is shown in the figures VII.3 and VII.5. The fluctuating velocity is on the order of several  $\frac{cm}{s}$ , whereas the mean velocity is usually less than 20% of the fluctuating velocity. We found that the mean velocity grows with increasing propeller speed. In the case of clockwise forcing, the fluctuating velocity grows linearly with the propeller frequency. In other words

$$\mathbf{u}' \propto f \quad (\text{VII.3})$$

This is in good agreement with results obtained by our group (e.g. [26]) in a French-Washing machine. We observe that counterclockwise forcing creates fewer turbulent fluctuations. Furthermore, random forcing is slightly more efficient in creating turbulent fluctuations than clockwise forcing at +500 rpm although it was designed to inject the same power into the fluid.

Spatial inhomogeneities in the flow could cancel each other and thus result in a small mean velocity. Therefore, the r.m.s. of the Eulerian mean velocity field was calculated. As depicted in figure VII.4, the spatial deviations are increasing with the motor frequency. However, the fluctuating velocities are still 3 to 4 times bigger than the r.m.s of the Eulerian velocity field.

No pronounced differences in the  $x$ ,  $y$  and  $z$  components of  $\mathbf{u}'$  are found except in the case of the anisotropic forcing. However, the flow tends to become more anisotropic with increasing propeller speeds. This might be because our propellers are self-made. Differences in the angle of the propeller vanes can be understood as forcing with different frequencies and thus a inhomogeneous and anisotropic injection of energy. As explained below, we found that the energy transfer rate is proportional to frequency cubed. Consequently, deviations between propellers will become detectable at sufficiently high frequencies.

We calculated the Eulerian second and third order structure function as is described in section VI.5. Taking the Kolmogorov constant  $C_2 = 2.1$ , together with the equations<sup>2</sup>(III.40), (III.39) and (III.43), enables us to compute three separate estimates of  $\varepsilon(r)$  with the corresponding error estimate  $\sigma_\varepsilon$ . For that purpose, we determine the inertial range,  $r_1 \dots r_2$ , from the width of the plateau of each structure function. The plateau region is typically only a few ten  $\eta$  long. Once the inertial range is known we compute

$$\varepsilon \approx \langle \varepsilon(r) \rangle \Big|_{r=r_1 \dots r_2} \quad \text{and} \quad \sigma_\varepsilon \approx \langle (\varepsilon(r) - \langle \varepsilon(r) \rangle)^2 \rangle^{1/2} \Big|_{r=r_1 \dots r_2}$$

---

<sup>2</sup>The equations are  $\varepsilon(r) \approx \left(\frac{D_{LL}}{C_2}\right)^{3/2} \frac{1}{r} = \left(\frac{3D_{NN}}{4C_2}\right)^{3/2} \frac{1}{r}$  and  $D_{LLL}(r, t) = -\frac{4}{5}\varepsilon r$ .

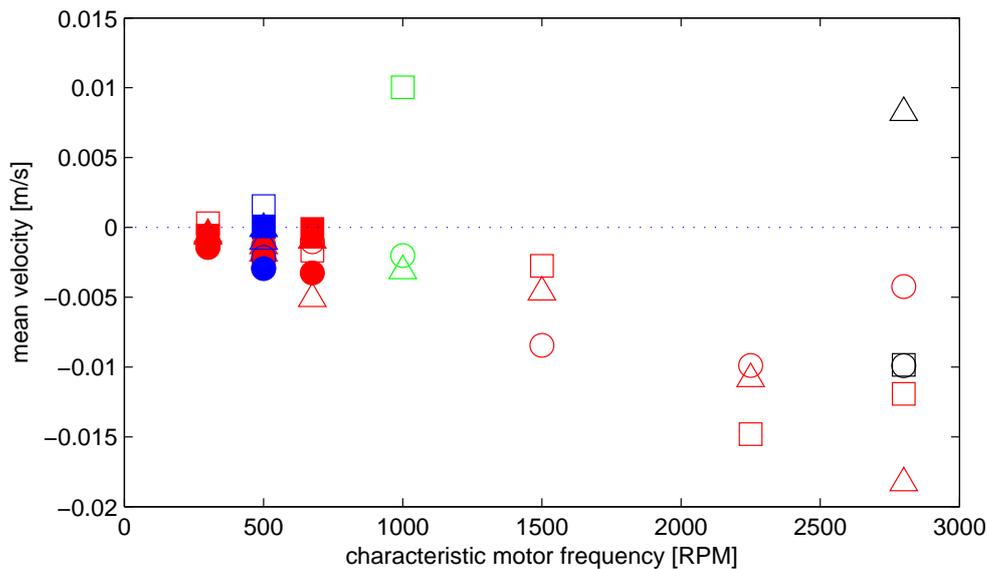


Figure VII.3: Mean velocity versus the motor frequency. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and **black**: *CCW* forcing. The symbols are:  $\triangle$  : *x*-component,  $\circ$  : *y*-component and  $\square$  : *z*-component. The runs with the small observation volume are marked with filled symbols. The mean velocity increases with the propeller speed and the difference between the three components of  $\langle \mathbf{u} \rangle$  become more distinct. However, the mean flow is less than 20% of the r.m.s of velocity.

for the second order structure functions, denoted  $\varepsilon(D_{LL})$  and  $\varepsilon(D_{NN})$ , and the third order structure function, denoted  $\varepsilon(D_{LLL})$ . As an example, the energy transfer rate,  $\varepsilon(r)$ , with clockwise forcing at +675 rpm is provided in figure VII.6. It should be noted that  $\varepsilon(D_{LLL})$  showed more scattering than  $\varepsilon(D_{LL})$  and  $\varepsilon(D_{NN})$ .

Figure VII.8 shows the energy dissipation rate for the experiments, we performed. The energy dissipation rate,  $\varepsilon$ , computed from the small measurement volumes were approximately 30% of their counterpart value acquired using the big observation volume. We found random forcing to be slightly more efficient than clockwise forcing at +500 rpm, and forcing at -2800 rpm to be less efficient than the propeller speed of +2800 rpm.

To verify that the difference between small and big observation volume runs is not due to the subtraction of the mean flow, we calculated the Eulerian second and third order structure functions without prior subtraction of the mean flow. As an example, the energy transfer rate,  $\varepsilon$ , with clockwise forcing at +675 rpm (bMV) is displayed in figure VII.7. The second order structure functions are comparable, whereas the third order structure function differs for separation distances bigger than approximately  $200 \eta$ . This fact also shows that the mean velocity field is stationary and small compared to the turbulent fluctuations.

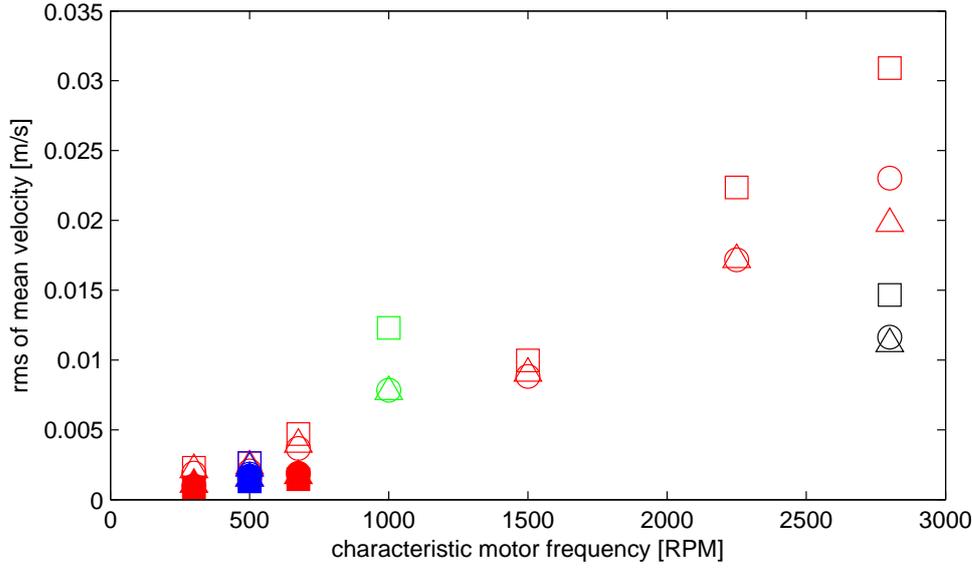


Figure VII.4: Spatial variations in the averaged Eulerian velocity field versus the motor frequency. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and **black**: *CCW* forcing. The symbols are:  $\Delta$  : *x*-component,  $\circ$  : *y*-component and  $\square$  : *z*-component. The runs with the small observation volume are marked with filled symbols. The mean flow becomes spatial inhomogeneous for increasing motor speed. The fluctuation velocity, however, are still 3 to 4-times stronger than the corresponding spatial inhomogeneity of the velocity field.

According to equation<sup>3</sup> (III.22) and the observation  $u' \propto f$ , it is

$$\varepsilon \propto f^3 \quad (\text{VII.4})$$

Therefore, the energy transfer rate should scale with the frequency as a power law with exponent 3. The dashed line in figure VII.8 is such a power law. In the big measurement volume the power law was observed for clockwise forcing. Figures VII.10 and VII.9 show the Kolmogorov time and length scales. According to the equations (III.25) and (III.23), they scale as

$$\eta \propto \varepsilon^{-1/4} \quad \Rightarrow \quad \eta \propto f^{-3/4} \quad (\text{VII.5})$$

and

$$\tau_\eta \propto \varepsilon^{-1/2} \quad \Rightarrow \quad \tau_\eta \propto f^{-3/2} \quad (\text{VII.6})$$

respectively. In both cases, we found good agreement between our measurements and the prediction.

In principle, it is possible to measure the integral length scale,  $L$ , using the Eulerian auto correlation and equation<sup>4</sup> III.48. However, since there is no analytical

<sup>3</sup>It is  $\varepsilon \approx \frac{u'^3}{L}$

<sup>4</sup> $L_{LL} \equiv \int_0^\infty f(r, t) dr$

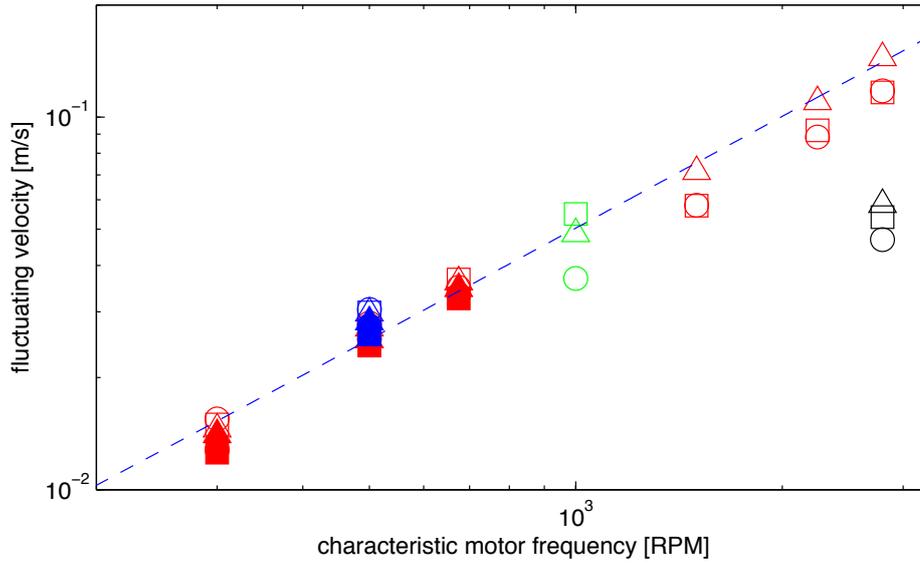


Figure VII.5: R.m.s of the velocity versus the motor frequency. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and black: *CCW* forcing. The symbols are:  $\triangle$  : *x*-component,  $\circ$  : *y*-component and  $\square$  : *z*-component. The runs with the small observation volume are marked with filled symbols. The dashed line shows a linear fit to the clockwise forcing (bMV). For clockwise and random forcing the r.m.s. of the velocity scales linear with the propeller frequency.

prediction on the autocorrelation functions, one needs to resolve a measurement volume with a size of several integral length scales. Therefore, we estimate  $L$  using equation (III.22):

$$L \approx \frac{u'^3}{\varepsilon} \quad (\text{VII.7})$$

The longitudinal Eulerian autocorrelation function,  $f(r)$ , showed an exponential decay over a region of several centimeter. We tried to fit an exponential to this region to estimate the integral length scale  $L$ . The different values of the integral length scale are displayed in figure VII.11. In the big measurement volume this method the estimates of  $L$  were approximately 4 cm, and showed only little dependence on the forcing. The values which we got from equation (VII.7), were a few times the value obtained from  $f(r)$ . Yet, the size of the big measurement volume is of the same order of magnitude than the integral length scale.

Equation (III.22) can be used also to calculate the Taylor scale Reynolds number,  $R_\lambda$ :

$$R_\lambda = \sqrt{15 \frac{u' L}{\nu}} = \left( \frac{15}{\nu} \right)^{1/2} \varepsilon^{-1/2} \langle u^2 \rangle \quad (\text{VII.8})$$

Figure VII.12 shows the estimates of the Taylor scale Reynolds number.  $R_\lambda$  ranging from approximately 150 to 380 has been achieved. No prediction on the dependence

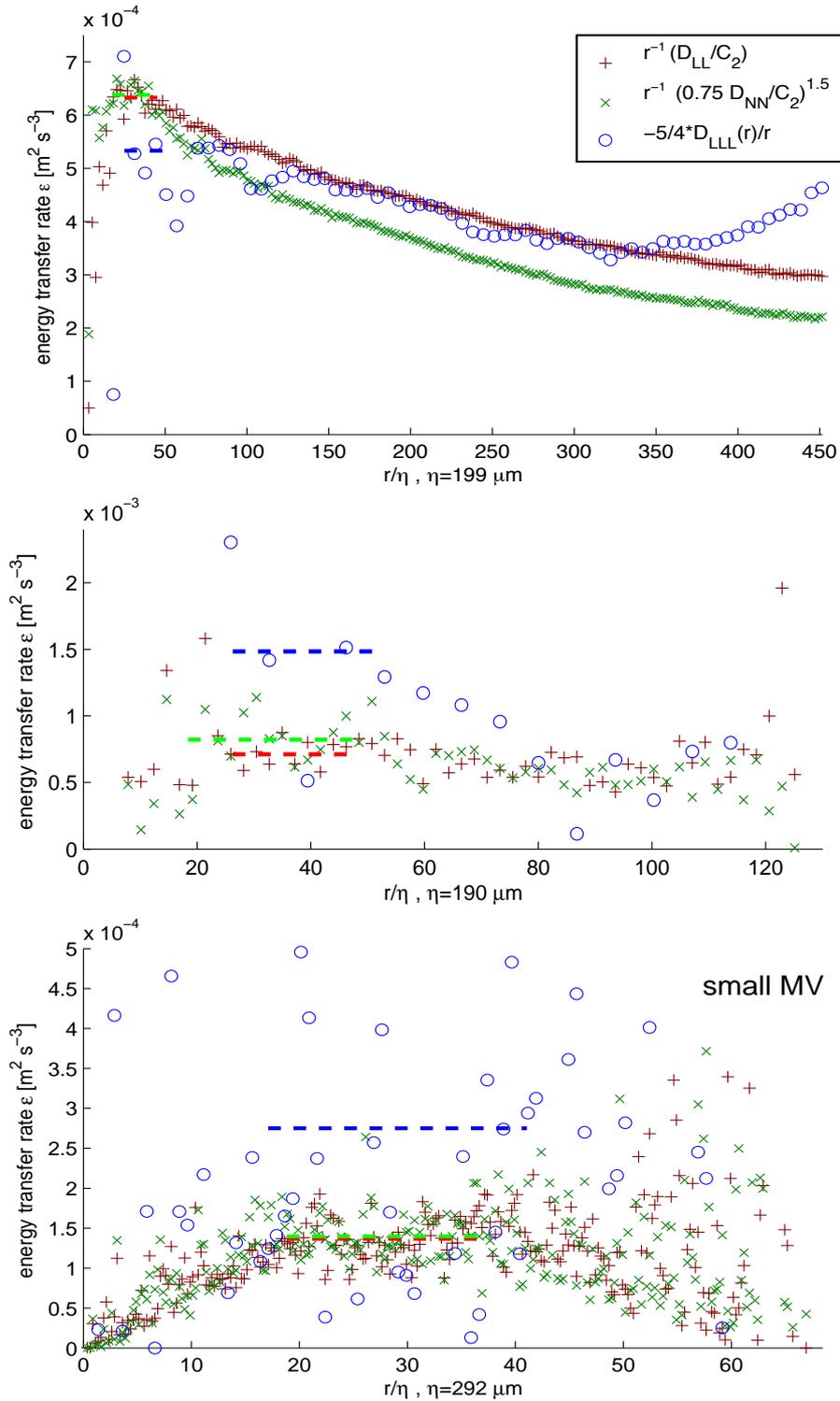


Figure VII.6:  $\varepsilon(r)$  at forcing with +675 rpm. Figure above: Energy transfer rate  $\varepsilon$  computed from the Eulerian structure functions at +675 rpm (bMV) forcing; figure in the middle: only points falling within a sphere with radius 1.2 cm where used for the calculation; the figure at the bottom: Energy transfer rate  $\varepsilon$  computed from the Eulerian structure functions for the same forcing but with the small measurement volume. The size of the small measurement volume is comparable to the size of the sphere. Although the forcing is the same,  $\varepsilon$  in the small measurement volume is approximately 30% of the value in the big measurement volume. However, the conditioned data (figure in the middle) does not show this behavior.

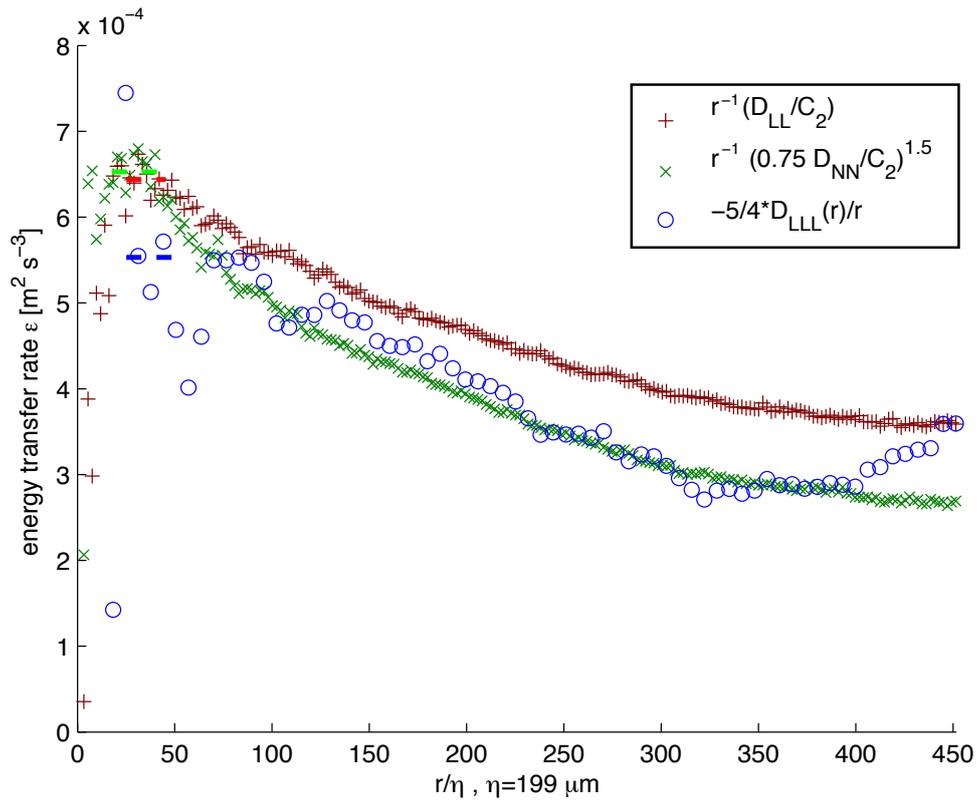


Figure VII.7:  $\varepsilon(r)$  at forcing with +675 rpm without subtracting the mean flow. The energy transfer rate computed from the second order structure function are comparable to the figure VII.6.  $\varepsilon(D_{LLL})$  is more affected by the mean flow field, but for separation distances up to  $200\eta$  still comparable. Thus, the difference between small measurement volume runs and big measurement volume runs is not due to the subtraction of the mean flow. Furthermore, it shows that the mean flow is small compared to the turbulent fluctuations in velocity.

on the frequency can be made although for an integral length scale independent of frequency  $R_\lambda$  should scale with  $f^{1/2}$ .

Table VII.1 and VII.2 provide a summary of the basic properties at different frequencies. The standard deviation in the components of  $\mathbf{u}'$  and  $\sigma_\varepsilon$  were used to estimate the errors of the listed measures.

At first glance, the flow of the Lagrangian Exploration Module is isotropic and the turbulence is three to ten times stronger than the mean flow. For propeller frequencies below 1000 rpm the mean velocity field shows only small spatial inhomogeneities. Thus, the flow is homogenous in this regime. Moreover, the energy transfer rate, as well as the Kolmogorov scales, measured for the big observation volume follow the predicted power laws. In addition, the big measurement volume is comparable to the integral length scale of the flow.

However, further investigation is needed.

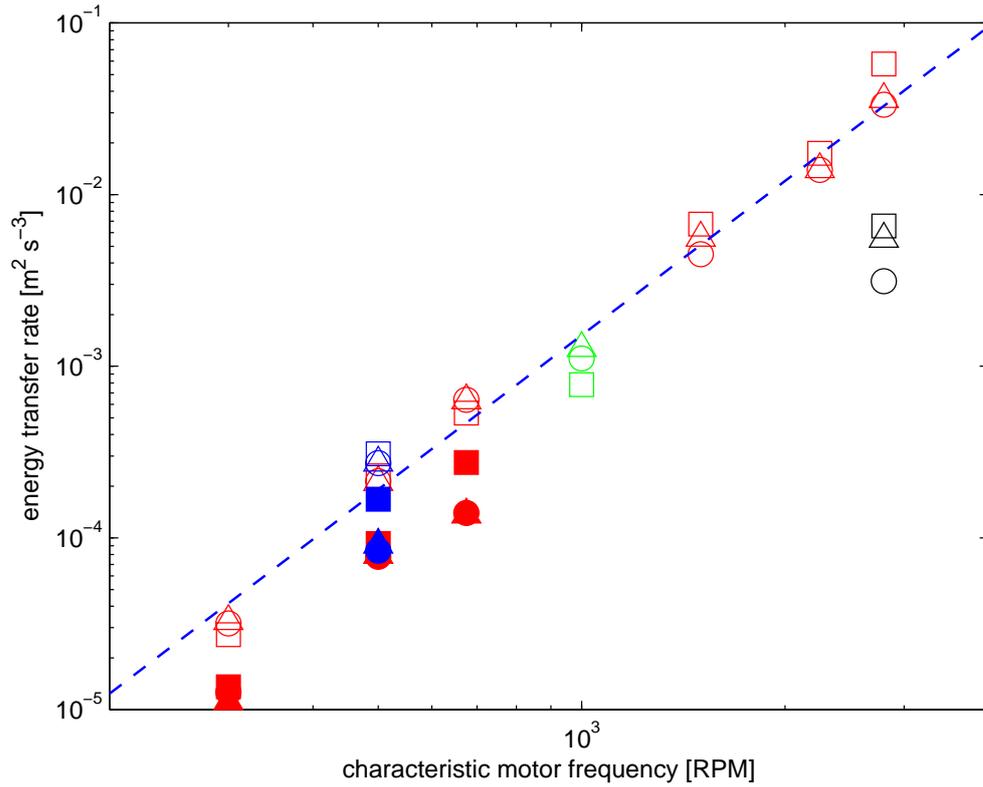


Figure VII.8: Energy transfer rate  $\varepsilon$  versus motor frequency.  $\varepsilon$  was independently calculated from  $D_{LL}$  ( $\triangle$ ),  $D_{NN}$  ( $\circ$ ) and  $D_{LLL}$  ( $\square$ ) using equations (III.40) and (III.43). Filled symbols mark the small observation volume runs. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and **black**: *CCW* forcing. The dashed line is power law fit to  $\varepsilon(D_{LL})$  and  $\varepsilon(D_{NN})$  at clockwise forcing in the big observation volume; the calculated exponent is 3.(0), which is in agreement with  $f \propto u'$  and  $\varepsilon \propto u'^3$ . The energy dissipation rate in the small measurement volume experiment is significantly smaller than for the big measurement volume runs.

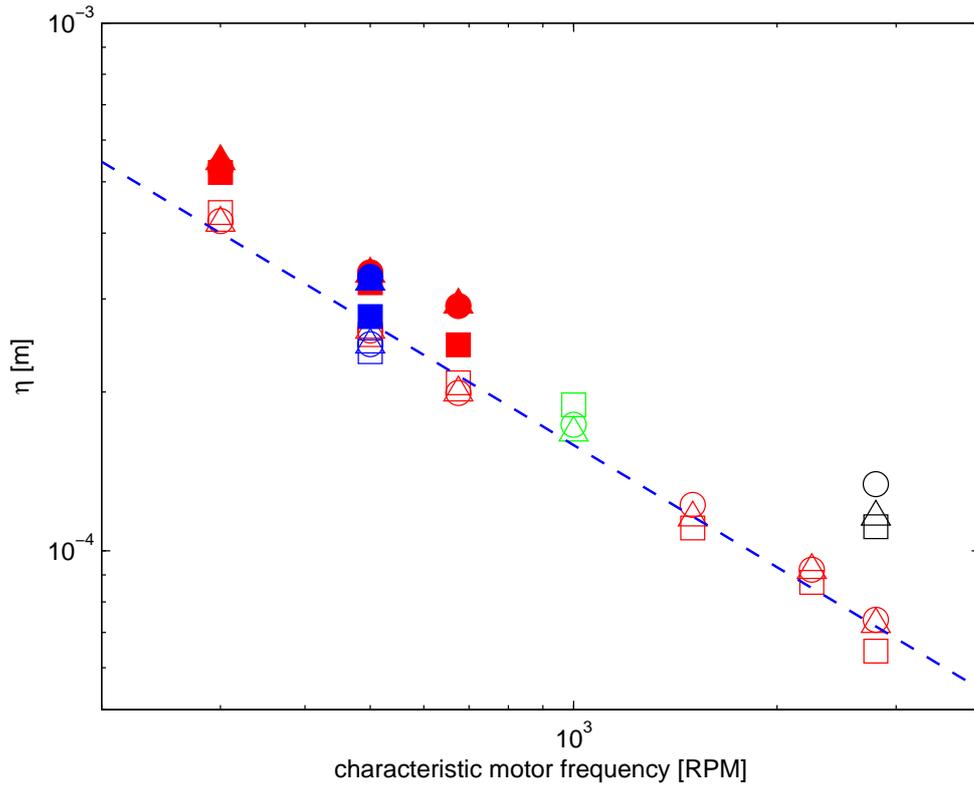


Figure VII.9: Kolmogorov length scale  $\eta$  versus motor frequency.  $\eta$  was calculated from  $\varepsilon(D_{LL})$  ( $\triangle$ ),  $\varepsilon(D_{NN})$  ( $\circ$ ) and  $\varepsilon(D_{LLL})$  ( $\square$ ). The small observation volume runs are marked with filled symbols. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and black: *CCW* forcing. The dashed line shows a power law fit to the clockwise forcing (bMV). Its exponent was determined as  $-0.7(7)$ . That is in good agreement with  $\eta \propto \varepsilon^{-1/4}$  and  $\varepsilon \propto f^3$ .

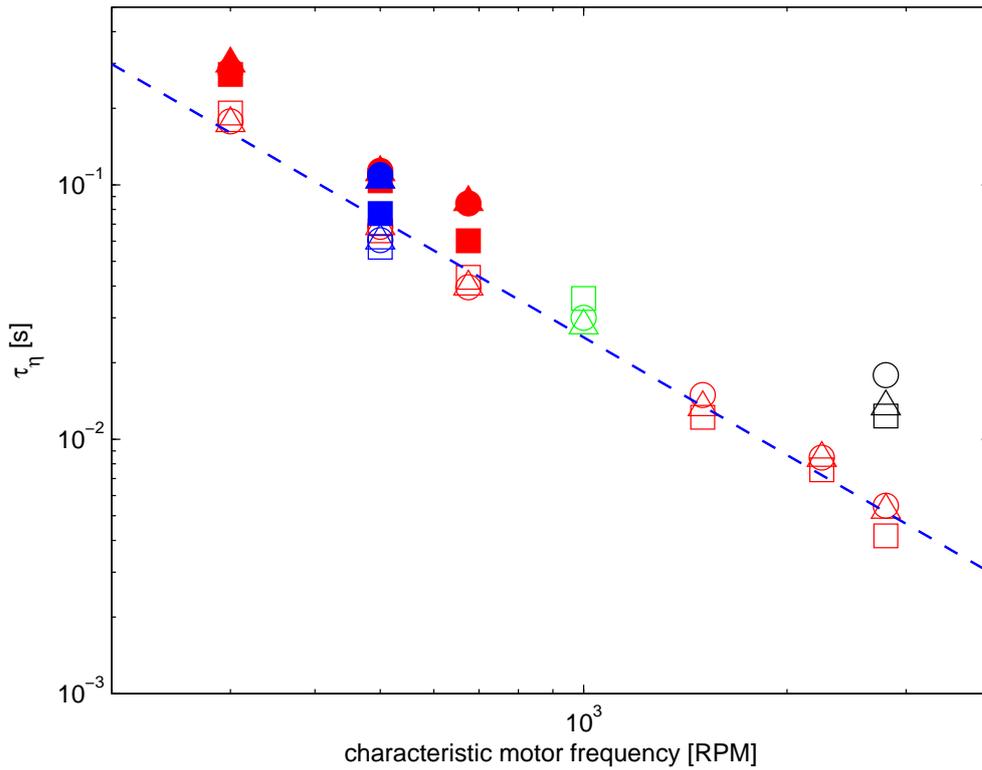


Figure VII.10: Kolmogorov time scale  $\tau_\eta$  versus motor frequency.  $\tau_\eta$  was calculated from  $\varepsilon(D_{LL})$  ( $\triangle$ ),  $\varepsilon(D_{NN})$  ( $\circ$ ) and  $\varepsilon(D_{LLL})$  ( $\square$ ). The small observation volume runs are marked with filled symbols. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and **black**: *CCW* forcing. The dashed line shows a power law fit to the clockwise forcing (bMV). Its exponent was determined as  $-1.5(4)$ . That is in good agreement with  $\eta \propto \varepsilon^{-1/2}$  and  $\varepsilon \propto f^3$ .

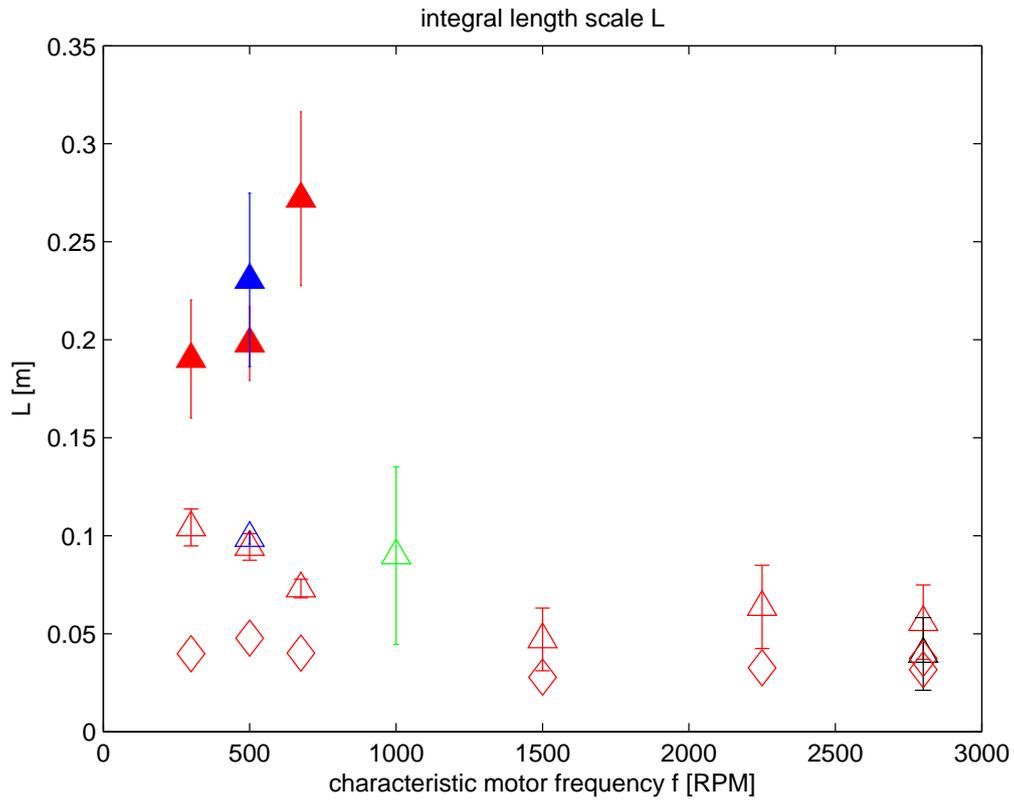


Figure VII.11: Integral length scale,  $L$ , versus motor frequency.  $L$  ( $\triangle$ ) was calculated from  $\varepsilon(D_{LL})$ ,  $\varepsilon(D_{NN})$ , and  $\varepsilon(D_{LLL})$  and their corresponding error estimates. The  $\diamond$ -symbol shows the estimate obtained the longitudinal Eulerian autocorrelation,  $f$ . The small observation volume runs are marked with filled symbols. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and black: *CCW* forcing. Because of  $L \propto \varepsilon^{-1}$  and the fact that energy transfer rate is smaller for the small measurement volume runs, the integral length scale of these experiments is bigger. The integral length scale has the order of magnitude of 10 *cm*. This is comparable to the size of the big measurement volume.

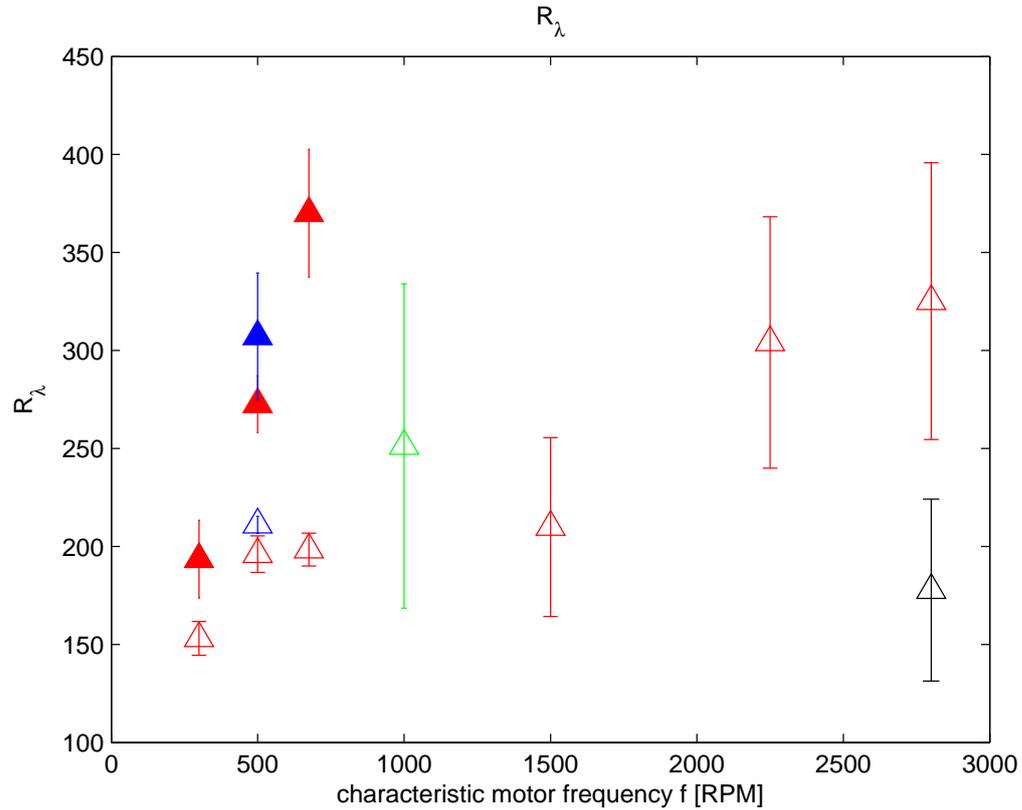


Figure VII.12: Taylor scale Reynolds number,  $R_\lambda$ , versus motor frequency.  $R_\lambda$  ( $\Delta$ ) was calculated from  $\varepsilon(D_{LL})$ ,  $\varepsilon(D_{NN})$ , and  $\varepsilon(D_{LLL})$  and their corresponding error estimates. The small observation volume runs are marked with filled symbols. The colors denote: **red**: *CW* forcing, **green**: *anisotropic* forcing, **blue**: *random* forcing and **black**: *CCW* forcing. The experiments covered a range from  $R_\lambda = 150$  to 380. Forcing at higher propeller speed should enable us to reach  $R_\lambda \approx 500$ .

Table VII.1: Energy transfer rate and Kolmogorov scales of the performed experiments

motor speed [rpm]	$\varepsilon$ [ $m^2 \cdot s^{-3}$ ]	$\eta$ [ $\mu m$ ]	$\tau_\eta$ [ms]	$R_\lambda$	$St$
300	$(3.24 \pm 0.15) \cdot 10^{-5}$	$420 \pm 5$	$175 \pm 4$	$150 \pm 10$	$4 \cdot 10^{-3}$
500	$(2.14 \pm 0.03) \cdot 10^{-4}$	$261 \pm 1$	$68 \pm 0.5$	$195 \pm 10$	$1.0 \cdot 10^{-2}$
675	$(6.3 \pm 0.1) \cdot 10^{-4}$	$200 \pm 2$	$40 \pm 1$	$200 \pm 10$	$1.7 \cdot 10^{-2}$
1500	$(5.2 \pm 0.75) \cdot 10^{-3}$	$120 \pm 4$	$14 \pm 1$	$210 \pm 45$	$4.9 \cdot 10^{-2}$
2250	$(1.4 \pm 0.2) \cdot 10^{-2}$	$91 \pm 4$	$8.4 \pm 0.6$	$300 \pm 60$	$8.5 \cdot 10^{-2}$
2800	$(3.6 \pm 0.5) \cdot 10^{-2}$	$73 \pm 3$	$5.3 \pm 0.3$	$330 \pm 70$	$13 \cdot 10^{-2}$
−2800	$(3.8 \pm 1.4) \cdot 10^{-3}$	$128 \pm 13$	$16 \pm 3$	$170 \pm 45$	$4.3 \cdot 10^{-2}$
300 (sMV)	$(1.14 \pm 0.02) \cdot 10^{-5}$	$540 \pm 8$	$290 \pm 8$	$190 \pm 20$	$5 \cdot 10^{-4}$
500 (sMV)	$(7.9 \pm 0.6) \cdot 10^{-5}$	$335 \pm 6$	$112 \pm 5$	$270 \pm 15$	$1.4 \cdot 10^{-3}$
675 (sMV)	$(1.4 \pm 0.2) \cdot 10^{-4}$	$290 \pm 10$	$85 \pm 6$	$370 \pm 30$	$1.8 \cdot 10^{-3}$
Anisotropic	$(1.14 \pm 0.17) \cdot 10^{-3}$	$170 \pm 7$	$30 \pm 3$	$250 \pm 80$	$2.4 \cdot 10^{-2}$
Random (sMV)	$(8.5 \pm 1.5) \cdot 10^{-5}$	$325 \pm 14$	$105 \pm 8$	$305 \pm 30$	$1.4 \cdot 10^{-3}$
Random (bMV)	$(2.74 \pm 0.03) \cdot 10^{-4}$	$245 \pm 1$	$60 \pm 1$	$210 \pm 5$	$1.2 \cdot 10^{-2}$

Table VII.2: Integral length scale and characteristic velocities of the performed experiments. The spatial inhomogeneities of the Eulerian mean velocity field are abbreviated as r.m.s. ( $\langle \mathbf{u} \rangle$ ).

motor speed [rpm]	$\langle \mathbf{u} \rangle$ [ $mm \cdot s^{-1}$ ]	$\mathbf{u}'$ [ $cm \cdot s^{-1}$ ]	r.m.s. ( $\langle \mathbf{u} \rangle$ ) [ $cm \cdot s^{-1}$ ]	L [cm]
300	(−0.6 −1.1 +0.3)	(1.45 1.54 1.50)	(0.2 0.2 0.2)	$10 \pm 1$
500	(−1.3 −2.9 +0.0)	(2.72 2.80 2.64)	(0.2 0.2 0.2)	$9 \pm 1$
675	(−5.1 −1.0 −1.7)	(3.61 3.49 3.66)	(0.4 0.4 0.5)	$7 \pm 1$
1500	(−4.6 −8.4 −2.8)	(7.15 5.79 5.78)	(0.9 0.9 1.0)	$5 \pm 1$
2250	(−10.8 −9.9 −14.8)	(11.0 8.8 9.2)	(1.7 1.7 2.2)	$6 \pm 2$
2800	(−18 −4.2 −11.9)	(14.4 11.8 11.6)	(1.9 2.3 3.1)	$6 \pm 2$
−2800	(+8.2 −9.9 −9.8)	(5.83 4.69 5.39)	(1.1 1.2 1.5)	$4 \pm 2$
300 (sMV)	(−0.5 −1.4 −0.6)	(1.40 1.27 1.26)	(0.1 0.1 0.1)	$19 \pm 3$
500 (sMV)	(−1.8 −1.3 −2.1)	(2.53 2.54 2.44)	(0.1 0.2 0.1)	$20 \pm 2$
675 (sMV)	(−0.9 −3.3 −0.1)	(3.47 3.33 3.26)	(0.2 0.2 0.1)	$27 \pm 5$
Anisotropic	(−3.1 −2.0 10)	(4.86 3.69 5.50)	(0.8 0.8 1.2)	$9 \pm 5$
Random (sMV)	(0.0 −3.0 0.0)	(2.82 2.76 2.61)	(0.2 0.2 0.1)	$23 \pm 5$
Random (bMV)	(−1.0 −2.1 +1.5)	(2.98 3.05 2.99)	(0.2 0.2 0.3)	$10 \pm 1$

## VII.4 Influence of the forcing

The LEM provides twelve propellers which can be controlled independently. It is worthwhile to investigate the influence of the forcing on the flow. As discussed in section VII.1 four types forcing were chosen to be tested. However, the quality of the data obtained only allows a detailed comparison between forcing at +500 rpm and random forcing in the big observation volume.

The random forcing was designed to ensure a constant energy injection rate. Since  $f \sim u'$  and  $\varepsilon \sim u'^3$ , we enforced

$$\frac{1}{12} \sum_{i=1}^{12} f_i^3 = f^3 = (500 \text{ rpm})^3 \quad \text{with } f_i \in [300 \text{ rpm}; 700 \text{ rpm}] \quad (\text{VII.9})$$

where  $f_i$  is the frequency of the  $i$ -th motor. Every 30 seconds new frequencies were randomly chosen. This time step is of the order of ten eddy turnover times.

### VII.4.1 Isotropy and homogeneity

One way to test the isotropy and homogeneity of the flow is provided by the Eulerian autocorrelation. Equation<sup>5</sup> (III.47) had been derived assuming an isotropic and homogeneous flow.

$$\begin{aligned} g(r) &= f(r) + \frac{r}{2} \left( \frac{\partial}{\partial r} f(r) \right) = \frac{1}{2r} \frac{\partial}{\partial r} (r^2 f(r)) \\ \Leftrightarrow f(r) &= \frac{2}{r^2} \int_0^r x g(x) dx \end{aligned} \quad (\text{VII.10})$$

The rearranged equation enables us to compute a prediction of  $f$  from the measured  $g$ . With the help of a discrete differentiation of  $f$ , a prediction of  $g$  is also possible. However, the prediction of  $g$  is more effected by noise than the prediction of  $f$ . For easier reading the predictions are denoted  $f_{th}$  and  $g_{th}$ , whereas the measured functions are called  $f_{exp}$  and  $g_{exp}$ .

We did observe that both  $\left| \frac{f_{th} - f_{exp}}{f_{exp}} \right|$  and  $\left| \frac{g_{th} - g_{exp}}{g_{exp}} \right|$  differ less than 15% for separation distances smaller between 3 cm to 5 cm for clockwise forcing. As an example, the Eulerian autocorrelation at forcing with +675 rpm (bMV) is depicted in figure VII.13. Furthermore, figure VII.13 shows that  $f$  seems to follow an exponential decay law for  $r < 9$  cm.

For further investigation on the isotropy, we plotted the dependence of the autocorrelation on the direction of the separation vector. With the help of spherical coordinates the direction,  $\mathbf{e}_r$ , was expressed as  $\mathbf{e}_r = \begin{pmatrix} \phi \\ \theta \\ r = 1 \end{pmatrix}$ . Figure VII.14 shows  $f$  and  $g$  for random forcing (bMV) conditioned on different values of  $\phi$  and  $\theta$ .

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<sup>5</sup>  $g(r) = f(r) + \frac{r}{2} \left( \frac{\partial}{\partial r} f(r) \right)$

Since the conditioned autocorrelation used less data than the total average, the first is more affected by measurement errors. However, both showed to be independent of the direction of the separation vector.

Thus, the flow shows good isotropy over ranges up to 8 *cm*. Again, this is on the order of the big measurement volume and also comparable to the integral length scale of the flow.

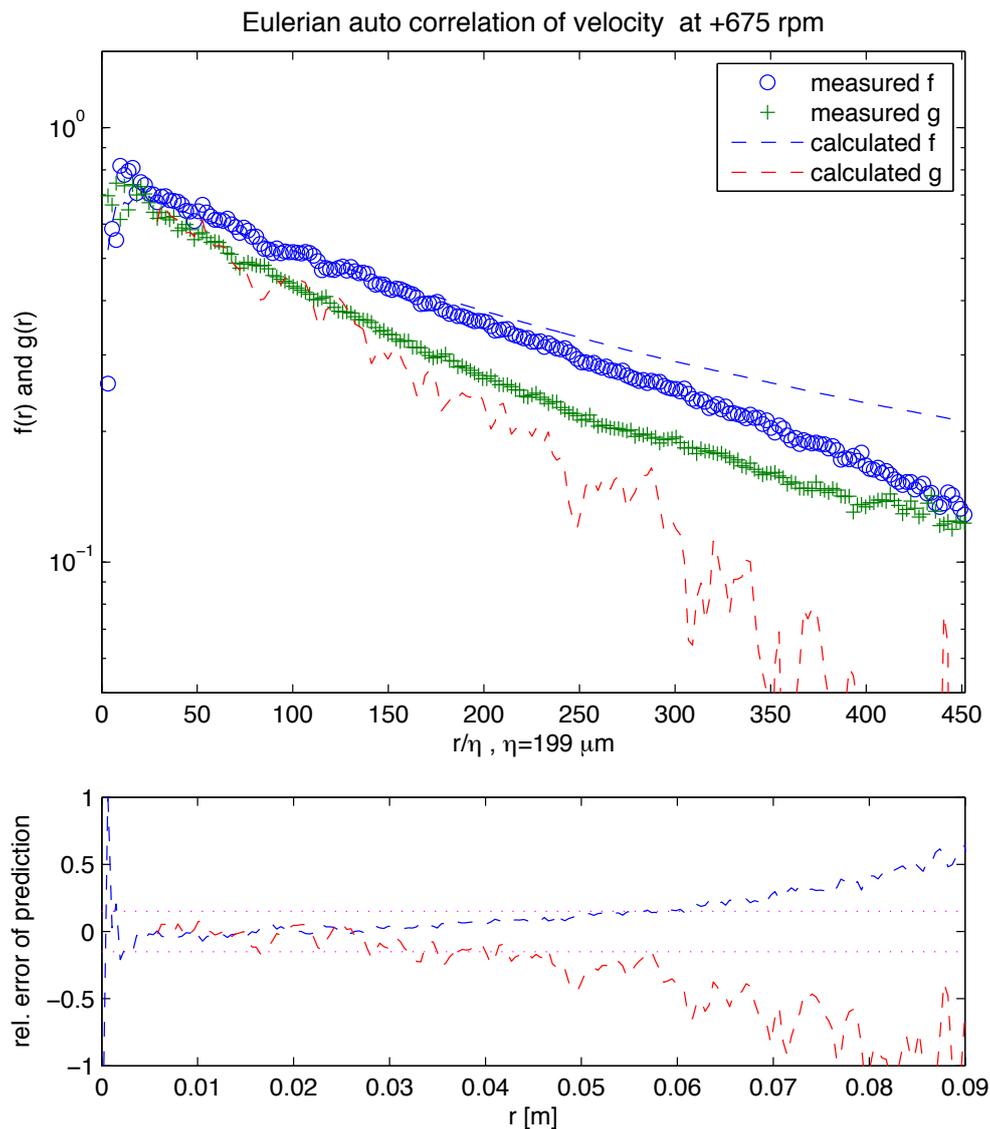


Figure VII.13: Eulerian autocorrelation at +675 *rpm* (bMV) forcing. The top figure shows the calculated and the measured autocorrelation curves  $f$  and  $g$ , whereas the figure at the bottom displays the relative deviation to the measured curves. The two dotted lines mark a region of  $\pm 15\%$  deviation. Because the curves fall within these bounds up to separations distances of approximately 4 *cm*, the flow in this regime can be considered locally isotropic.

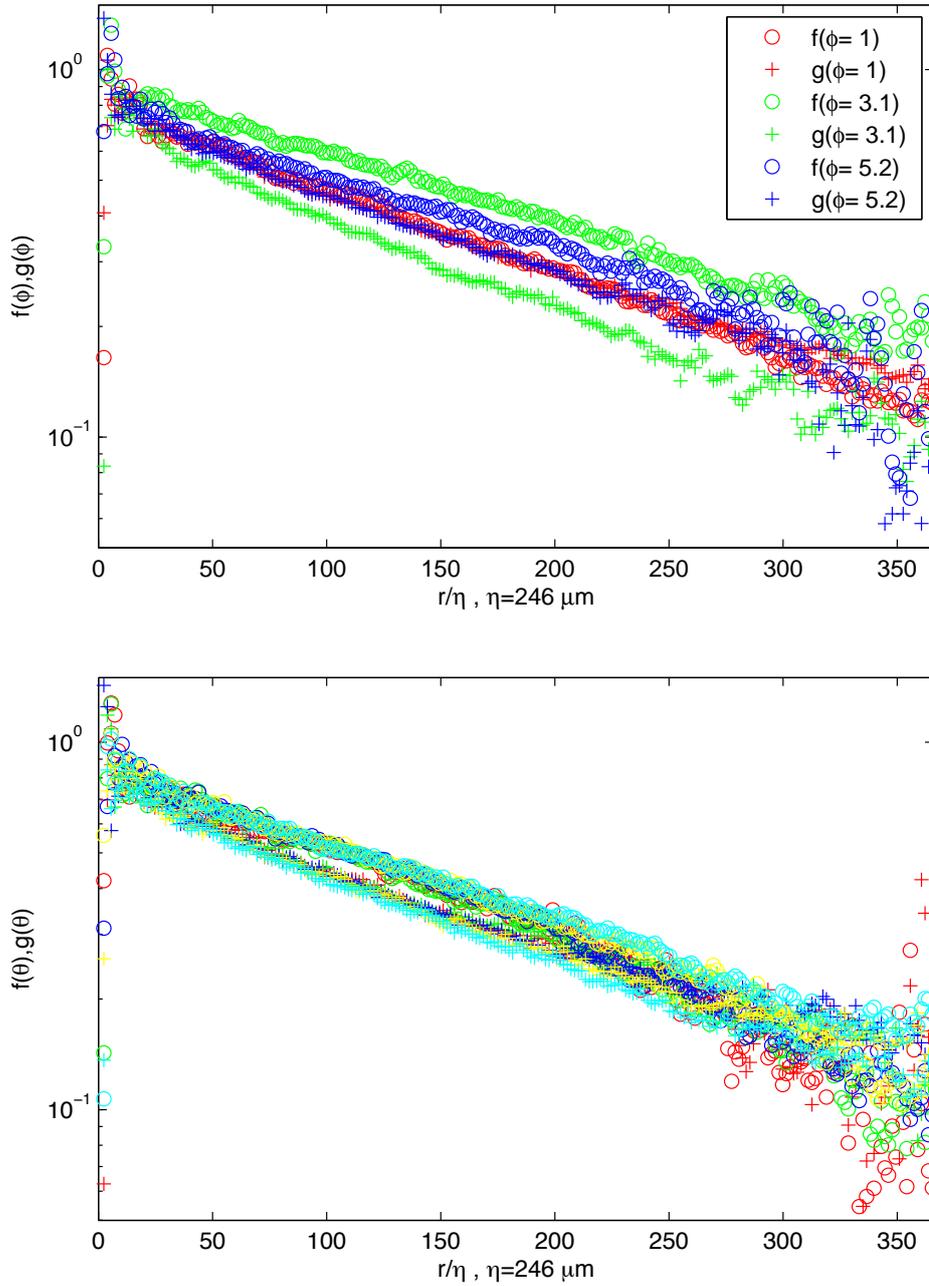


Figure VII.14: Eulerian autocorrelation conditioned on direction at random (bMV) forcing. The Eulerian autocorrelation was conditioned on the direction of the separation vector  $\mathbf{r}$  in spherical coordinates  $(\phi, \theta)$ . The top figure shows the dependence of  $f$  and  $g$  on  $\phi$ , and the bottom figure plots the dependence of  $f$  and  $g$  on  $\theta$ . In the latter figure, five bins for  $\theta$  were used and the colors denote red:  $\theta \in [1.37|1.57]$ , green:  $\theta \in [1.16|1.37]$ , blue:  $\theta \in [0.92|1.16]$ , yellow:  $\theta \in [0.64|0.92]$ , cyan:  $\theta \in [0|0.64]$ . Differences between the curves are probably due to the reduced amount of used data points per curve. However, the Eulerian autocorrelation functions  $f$  and  $g$  are independent of the direction. Thus, the Lagrangian exploration module creates isotropic turbulence.

## VII.4.2 Probability density functions

PDFs of velocity and acceleration were measured and compared. In good agreement with [24], the measured acceleration PDF from the big observation volume show the same shape as the acceleration PDF measured with the silicon strip detector experiment [25] in their French-Washing-Machine. Furthermore, we noticed that for the big measurement volume the shape of the acceleration PDF did not depend on the forcing. As an example, the acceleration PDF of +500 rpm in the big measurement volume is provided in figure VII.15.

However, the PDFs of velocity were not gaussian. Their shape is comparable to the axial velocity PDF obtained in a French-Washing machine [26]. Furthermore, we noted minor changes in the velocity PDF due to the forcing. Figure VII.17 and VII.16 display the velocity PDF of anisotropic forcing and +500 rpm forcing (bMV). In the anisotropic case,  $y$  and  $z$ -component of the velocity distribution seem to follow  $pdf(u - \langle u \rangle) \propto \exp(-\|u - \langle u \rangle\|)$ . The curves of the +500 rpm run are smoother.

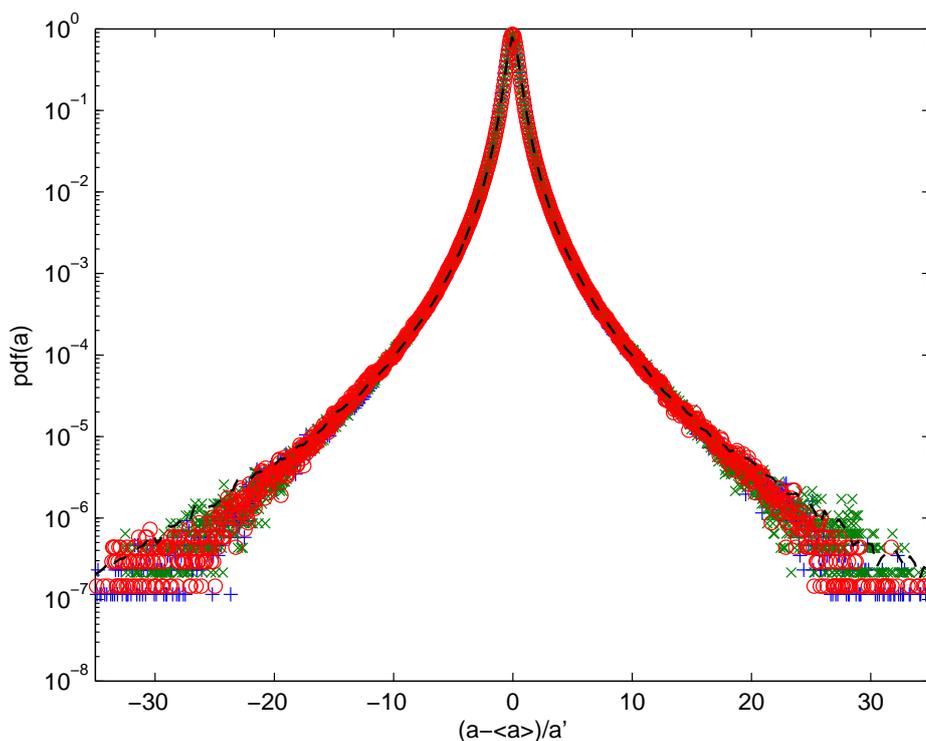


Figure VII.15: Probability density function of the acceleration for the +500 rpm (bMV) run. The symbols denote +  $x$ -component,  $\times$   $y$ -component and  $\circ$   $z$ -component. The dashed curve is the acceleration PDF ( $R_\lambda = 690$ ) published in [24]. The less distinct tails of the acceleration in the LEM are probably due to smaller temporal used.

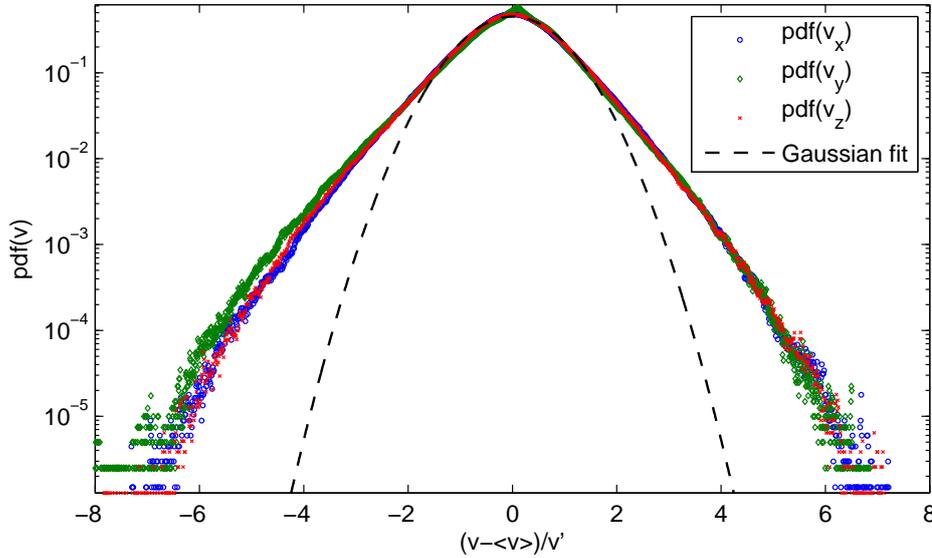


Figure VII.16: Probability density function of the velocity for the +500 rpm (bMV) run. The PDF of velocity is clearly not Gaussian. Unlike the velocity p.d.f of anisotropic forcing (see figure VII.17), the distribution at the  $u \approx \langle u \rangle$  is smooth.

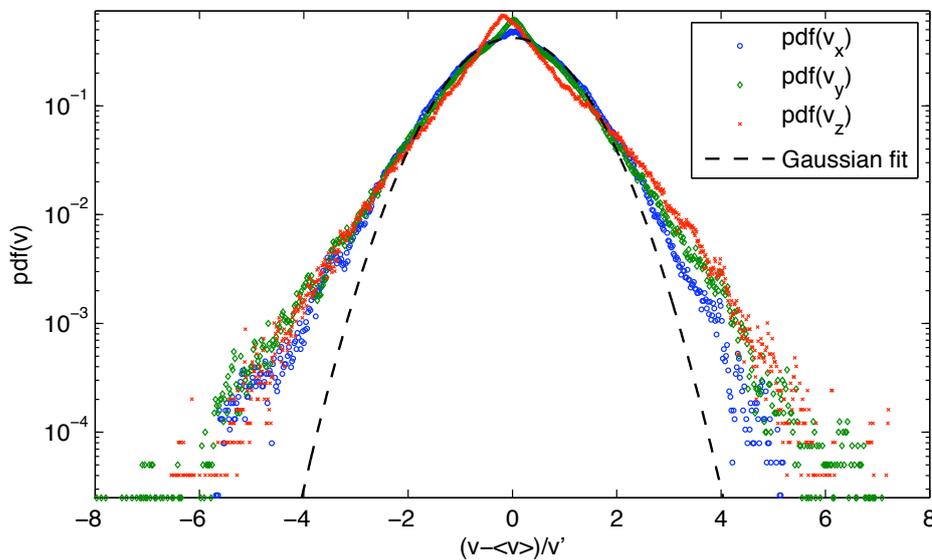


Figure VII.17: Probability density function of the velocity for the anisotropic run. No Gaussian velocity distribution was observed, moreover the PDF of each velocity component shows a kink at the  $\langle u \rangle$ . Moreover,  $y$  and  $z$ -component of the velocity distribution seem to be proportional to  $\exp(-\|u - \langle u \rangle\|)$ .

### VII.4.3 Mean-flow field

The Eulerian mean velocity field has been measured as described in section VI.3. For small distances between  $\mathbf{r}_0$  and  $\mathbf{r}$  we can approximate the mean flow,  $\mathbf{u}(\mathbf{r})$ , as

$$\mathbf{u}(\mathbf{r}) \approx \mathbf{u}_0 + \underline{\underline{\mathbf{M}}}(\mathbf{r} - \mathbf{r}_0) \quad (\text{VII.11})$$

where  $\underline{\underline{\mathbf{M}}}$  is the Jacobian<sup>6</sup>,  $\nabla \mathbf{u}(\mathbf{r}_0)$ , of the velocity field at  $\mathbf{r}_0$ . The fitting procedure described in section VI.4 allows us to compute the Jacobian  $\underline{\underline{\mathbf{M}}}$  and  $\mathbf{u}_0$ . All points lying within a sphere of radius 5 cm, which was located at the center of the measurement volume, were included in the fitting process.

The Jacobian contains two physical properties: the symmetric part of  $\underline{\underline{\mathbf{M}}}$  is the rate of strain tensor

$$S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (\text{VII.12})$$

whereas the antisymmetric part contains the rotation tensor

$$\Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad (\text{VII.13})$$

of the fluid.

$\sigma = (\sigma_1 \mid \sigma_2 \mid \sigma_3)$  denotes the three eigenvalues<sup>7</sup> of the rate of strain tensor  $\underline{\underline{\mathbf{S}}}$ . The continuity equation dictates

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \quad (\text{VII.14})$$

Consequently, the rate of strain is fully described by two of its eigenvalues and their corresponding eigenvectors. We will use sorted eigenvalues, i.e.

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \quad (\text{VII.15})$$

The local vorticity of the flow is defined as:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \nabla \times \mathbf{u} \quad (\text{VII.16})$$

Expressing  $\boldsymbol{\omega}$  in terms of  $\Omega_{ij}$  it is

$$\omega_1 = 2\Omega_{32}, \quad \omega_2 = 2\Omega_{13} \quad \text{and} \quad \omega_3 = 2\Omega_{21} \quad (\text{VII.17})$$

For the forcing at +500 rpm and random forcing (both big observation volume runs), we stored the mean flow field for each trajectory file. This enables us to investigate the rate of strain ( $\sigma_1$  and  $\sigma_2$ ), the local vorticity ( $\omega_1, \omega_2, \omega_3$ ) and  $\mathbf{u}_0$  with respect to their evolution with increasing number of collected data points.

<sup>6</sup>or velocity-gradient tensor

<sup>7</sup>In order to avoid confusion with the Taylor micro scale,  $\sigma$  instead of  $\lambda$  will be used to denote the eigenvalues of the rate of strain tensor.

Because the observation time is only a few seconds but the time to download (and process) the raw movies is several minutes, it is not meaningful to use the real time for investigating the evolution. Therefore and also because the averaging process depends only on the number of data points but not on real time, the number data points collected was chosen as a measure of time.

Figures VII.18, VII.19 and VII.20 show the convergence of the investigated properties for an increasing amount of collected data. The three plots show that all of the properties approach their mean value after approximately  $2 \cdot 10^7$  data points. This corresponds to approximately 1 GB to 3 GB of raw data. After reaching this first point, they change much slower or settle.

In both experiments  $\mathbf{u}_0$  differed from  $\langle \mathbf{u} \rangle$ <sup>8</sup> by less than one  $\frac{mm}{s}$  in each component. This corresponds to a spatial homogeneous flow within the sphere used in the fitting procedure.

We observe, that all the three eigenvalues of the rate of strain tensor are non-negligible. If one eigenvalue is negligible, this corresponds to a flow which is stretched in (exactly) one direction and contracted along (exactly) one direction perpendicular to the first. Thus, we observe a three-dimensional mean flow field and no two-dimensional flow pattern. Furthermore, we find that  $\sigma_2$  and  $\sigma_3$  are negative with both forcings.

The magnitude of the rate of strain VII.19 and the local vorticity VII.20 are both on the order of several  $\frac{1}{100} \frac{1}{s}$ ; whereas random forcing results in a less rotating mean flow. Also we note that  $\sigma_1(\text{Random forcing}) < \sigma_1(+500 \text{ rpm forcing})$ . However, the difference between the both is small.

It is also possible to measure the rate of strain, rotation and velocity by using the fitting method to determine the velocity gradient and  $\mathbf{u}_0$  on a small subset of the data. We divided the time measured (characterized by the number of data points) into 50 consequent subsets which contain approximately the same amount of data points. Each subset corresponds to 1 to 2 hours of measurement time. They enable us to compare snapshots of the mean flow to the long time averaged flow field.

Figure VII.22 displays the short-time-averaged  $|\mathbf{u}_0|$  for forcing with +500 rpm and random forcing. Both forcings have the same magnitude and range from 2 to 10  $\frac{mm}{s}$ , which is several times bigger than their corresponding value obtained from the long time averaged velocity field. The magnitude of the local vorticity,  $|\omega|$ , calculated from the short-time averaged velocity field, is provided in figure VII.21. The behavior of  $|\omega|$  is comparable to  $|\mathbf{u}_0|$ : both forcings have the same magnitude, which is between 0.05 and 0.25  $\frac{1}{s}$ . That value, however, is several times bigger than  $|\omega|$  obtained from the long time average of the Eulerian velocity field.

The short-time averaged eigenvalues of the rate of strain tensor are depicted in figure VII.23. The magnitude of the rate of strain is approximately 2 to 3 times bigger than the corresponding value of the long time averaged mean field. Surprisingly, the second eigenvalue<sup>9</sup>,  $\sigma_2$ , of the rate of strain tensor fluctuates from close to zero to positive values. The  $\sigma_2$  computed from the long time average velocity field showed

<sup>8</sup>calculated from the velocity PDFs

<sup>9</sup>The eigenvalues are sorted, i.e.  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ . Thus,  $\sigma_1 > 0$

$\sigma_2 < 0$ . If  $\sigma_1$  and  $\sigma_2$  are positive then one can picture a sphere which is stretched to a pancake-like shape. Its volume does not change due to the continuity equation. If  $\sigma_2$  and  $\sigma_3$  are both negative, the sphere mentioned before would be deformed to cigar shaped body. It is worthwhile to note that Xu *et al* [27] observed that the instantaneous rate of strain tensor in intense turbulence shows to two positive eigenvalues.

For a better comparison, we provide  $\frac{\sigma_2}{\sigma_1}$  and  $\frac{\sigma_3}{\sigma_1}$  in figure VII.23. Although we observed that random forcing shows  $\frac{\sigma_2}{\sigma_1} \approx \frac{\sigma_3}{\sigma_1}$  for the long time averaged mean flow field, we found  $\left| \frac{\sigma_2}{\sigma_1} \right| \approx 0$  and consequently  $\frac{\sigma_3}{\sigma_1} \approx -1$  in the case of short-time averaged measurements.

The short-time averaged velocity field show no difference between forcing with +500 rpm and random forcing. Thus, it is either not possible to determine differences between both forcings with the measurement techniques employed or the mean flow does not change for small perturbations to its creation. The latter would coincide with Kolmogorov idea that the turbulence is independent of the forcing and only depending on the energy transfer rate for small regions far away from the boundaries and singularities of the flow.

However, the long-time averaged velocity field reveals that random forcing creates a less rotating mean flow field. It could be that forcing with +500 rpm creates a big, slowly rotating whirl in the LEM which is less pronounced with random forcing. For a further investigation of the Eulerian flow field, either particle tracking velocimetry with much higher seeding densities or tomo-PIV should be considered as possible measurement techniques.

Six dimensionless numbers quantities can be build with the rate of strain and the vorticity, which relate the change of the mean flow to the Kolmogorov time scale, the frequency of the propellers or the eddy turn over time<sup>10</sup>,  $T$ :

$$A_{\tau,\sigma} \equiv \tau_\eta \cdot \max(\|\sigma_1\|, \|\sigma_3\|) \quad (\text{VII.18a})$$

$$A_{\tau,\omega} \equiv \tau_\eta \cdot \|\langle \omega \rangle\| \quad (\text{VII.18b})$$

$$A_{f,\sigma} \equiv f^{-1} \cdot \max(\|\sigma_1\|, \|\sigma_3\|) \quad (\text{VII.18c})$$

$$A_{f,\omega} \equiv f^{-1} \cdot \|\langle \omega \rangle\| \quad (\text{VII.18d})$$

$$A_{T,\sigma} \equiv T \cdot \max(\|\sigma_1\|, \|\sigma_3\|) \quad (\text{VII.18e})$$

$$A_{T,\omega} \equiv T \cdot \|\langle \omega \rangle\| \quad (\text{VII.18f})$$

where  $f$  is the characteristic frequency of the propellers. If the mean flow is stationary, i.e. changing much slower than the characteristic time scale of the turbulent fluctuations,  $A_{\tau,\omega}$  and  $A_{\tau,\sigma}$  are much smaller than unity. Furthermore, the idea of scale separation yields that  $A_\omega$  and  $A_\sigma$  should decrease with higher Reynolds numbers.

If  $A_{T,\sigma}$  and  $A_{T,\omega}$  are much smaller than 1, then the mean-flow changes slowly compared to the time scale of the biggest eddies.

<sup>10</sup>The eddy turn over time is the characteristic time scale of the biggest eddies, and defined as  $T \equiv \frac{L}{u}$

The dimensionless numbers,  $A_{f,\sigma}$  and  $A_{f,\omega}$ , test if the time scale of the flow field is related to the propeller speed.

Figure VII.24 and table VII.3 show these six properties for the experiments performed. To determine the local vorticity and the rate of strain for the small measurement runs, the size of the fitting volume was set to 1.5 cm.

The mean flow characterized by rate of strain,  $\sigma$ , and local vorticity,  $\omega$ , changes 5 to 10 times slower than the eddy turn over time. Consequently,  $A_{\tau,\sigma}$  and  $A_{\tau,\omega}$ , are ranging from  $10^{-3}$  to  $10^{-2}$ . In other words, the mean flow is changing a few hundred to one thousand times slower than the turbulent fluctuations characterized by the Kolmogorov time scale. We observe that the dimensionless numbers based on the Kolmogorov time scale decrease with an increase in motor speed ( and thus also  $R_\lambda$ ). This supports Kolmogorov's idea of scale separation, i.e. with intenser turbulence smaller Kolmogorov scales will be reached, whereas the big scales remain unchanged. However, this trend is small and probably within the uncertainty of  $A_{\tau,\sigma}$  and  $A_{\tau,\omega}$ .

The parameters based on propeller speed and rate of strain and vorticity are approximately  $\frac{1}{100}$  but not unity. Therefore, the timescale of the mean flow is not on the order of the propeller speed. Moreover,  $A_{f,\sigma}$  and  $A_{f,\omega}$ , change only little with the frequency of the propellers. More measurements with a higher accuracy would be needed to provide a better description.

In general, we noted that random forcing (bMV) creates a less rotating flow than the other forcing investigated.

Table VII.3: Time scales of the mean flow compared to the Kolmogorov time scale

motor speed $f_c$ [rpm]	$A_{\tau,\sigma}$	$A_{\tau,\omega}$	$A_{f,\sigma}$	$A_{f,\omega}$	$A_{T,\sigma}$	$A_{T,\omega}$
300	$1 \cdot 10^{-2}$	$7 \cdot 10^{-3}$	$13 \cdot 10^{-3}$	$9 \cdot 10^{-3}$	0.46	0.30
500	$5 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$8 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	0.24	0.10
675	$4 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	$9 \cdot 10^{-3}$	$10 \cdot 10^{-3}$	0.21	0.23
1500	$3 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	$9 \cdot 10^{-3}$	$12 \cdot 10^{-3}$	0.17	0.23
2250	$4 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$15 \cdot 10^{-3}$	$11 \cdot 10^{-3}$	0.37	0.28
2800	$4 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$18 \cdot 10^{-3}$	$12 \cdot 10^{-3}$	0.37	0.25
-2800	$1 \cdot 10^{-2}$	$3 \cdot 10^{-3}$	$15 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	0.54	0.16
300	$2 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	$17 \cdot 10^{-3}$	$23 \cdot 10^{-3}$	1.2	1.7
500	$1 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$14 \cdot 10^{-3}$	$12 \cdot 10^{-3}$	0.9	0.8
675	$1 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$12 \cdot 10^{-3}$	$15 \cdot 10^{-3}$	1.1	1.4
Anisotropic	$2 \cdot 10^{-2}$	$8 \cdot 10^{-3}$	$3 \cdot 10^{-2}$	$16 \cdot 10^{-3}$	1.14	0.51
Random (sMV)	$1 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$18 \cdot 10^{-3}$	$16 \cdot 10^{-3}$	1.2	1.1
Random (bMV)	$3.7 \cdot 10^{-3}$	$1.2 \cdot 10^{-2}$	$10 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	0.27	0.05

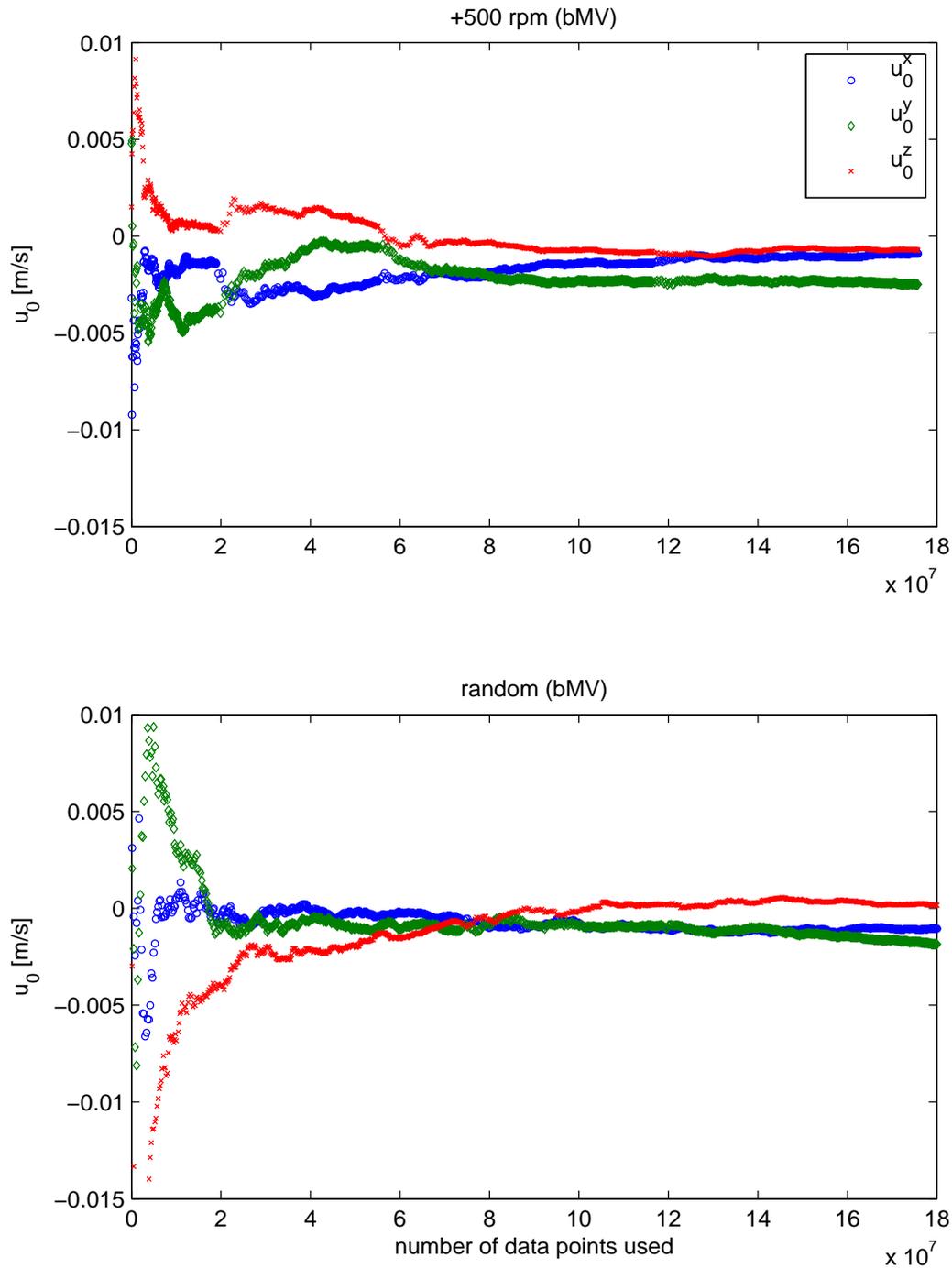


Figure VII.18: Evolution of  $\mathbf{u}_0$  with the number data points measured. The velocity,  $\mathbf{u}_0$ , approaches its mean value after approximately  $2 \cdot 10^7$  data points used. This corresponds to approximately 1 GB to 3 GB of raw data. After reaching this first point  $\mathbf{u}_0$  changes much slower to a final velocity on the order of a  $\frac{mm}{s}$ . That value agrees well with the mean velocity obtained from the PDF of  $\mathbf{u}$  (see section VII.3).

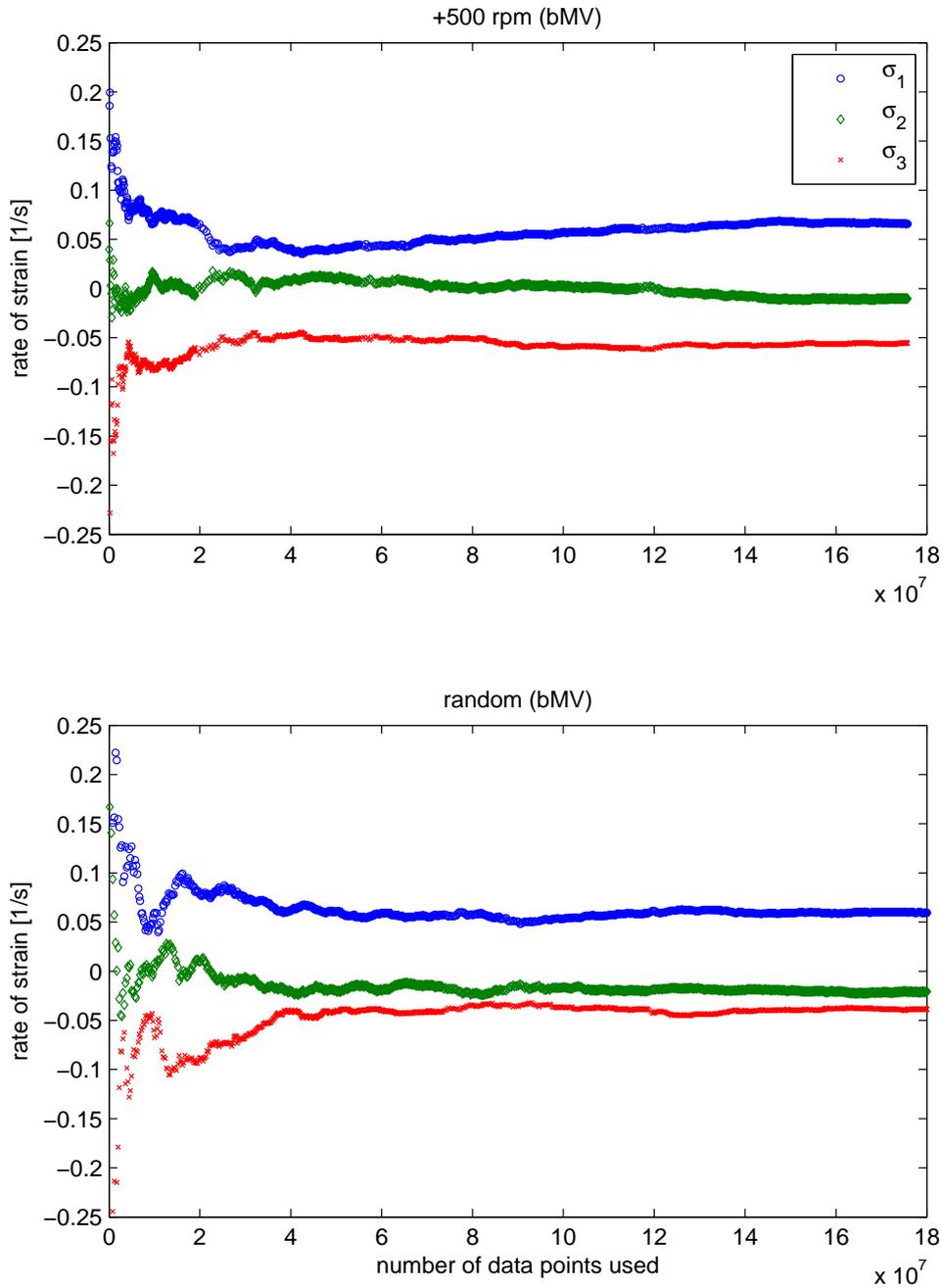


Figure VII.19: Evolution of rate of strain with the number data points measured. The rate of strain characterized by its eigenvalues  $\sigma_1, \sigma_2, \sigma_3$ , is approaching a value on the order of  $\frac{1}{20s}$  for both random forcing and forcing with  $+500 \text{ rpm}$ . Also,  $\sigma_2$  and  $\sigma_3$  are both negative. However, random forcing creates a rate of strain tensor whose  $\frac{\sigma_2}{\sigma_1}$  is comparable to  $\frac{\sigma_3}{\sigma_1}$ . In the case of forcing with  $+500 \text{ rpm}$ ,  $\left| \frac{\sigma_2}{\sigma_1} \right|$  is small compared to  $\left| \frac{\sigma_3}{\sigma_1} \right|$ . Thus, random forcing creates flow which stretched and compressed along three directions, whereas forcing with  $+500 \text{ rpm}$  results in a flow which is stretched and compressed mainly along two directions. It should also be noted, that the curve change slower after approximately  $3 \cdot 10^7$  data points used, which is in agreement with figure VII.18.

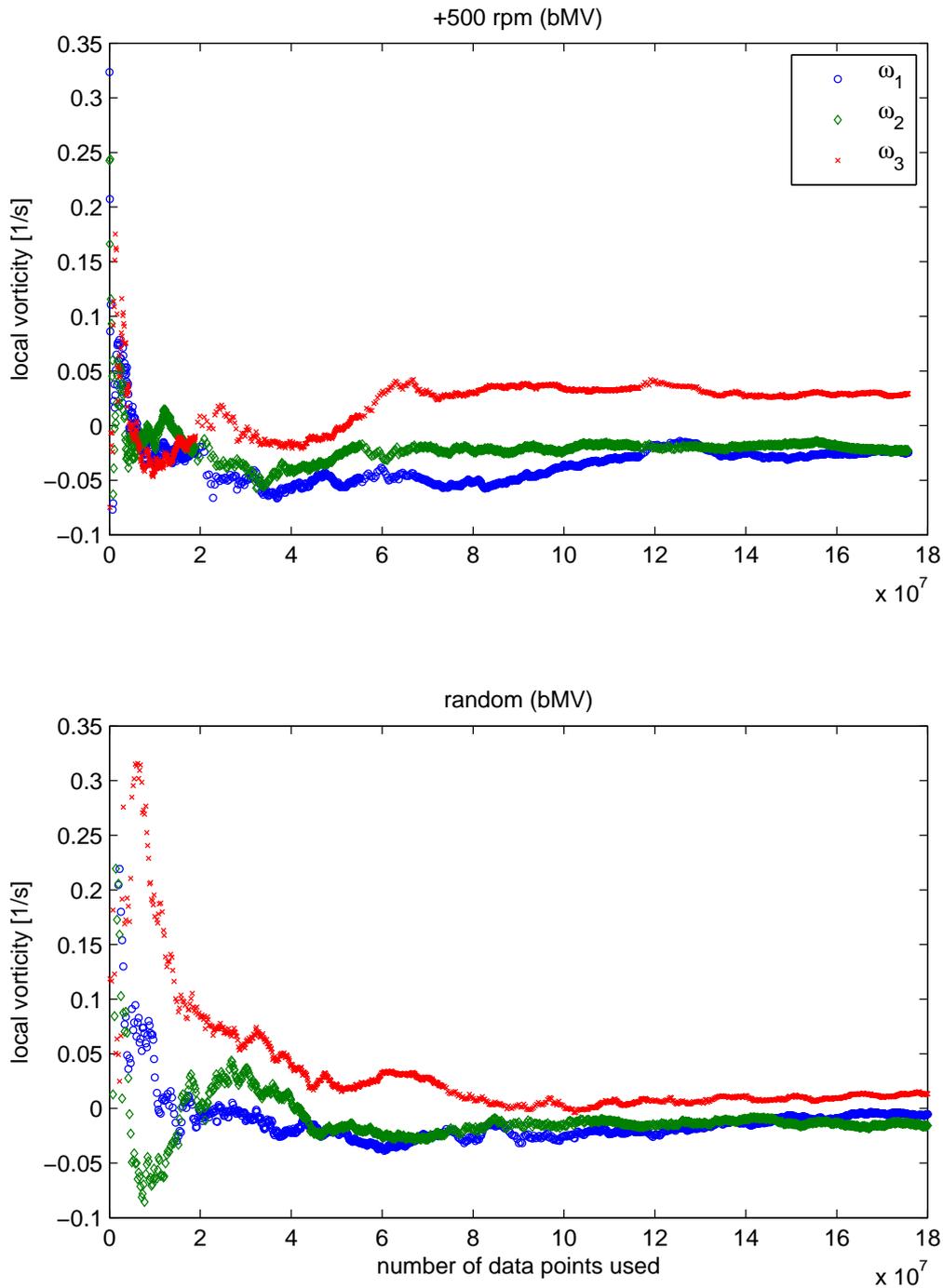


Figure VII.20: Evolution of the local vorticity with the number data points measured. The local vorticity,  $\omega$ , is approaching a frequency on the order a few  $\frac{1}{100}$  Hz. Random forcing shows to produce more fluctuations in the local vorticity than forcing with +500 rpm if less than  $8 \cdot 10^7$  data points were used. However, the average local vorticity is bigger for forcing with +500 rpm. In other words, forcing at +500 rpm creates a more stable and pronounced rotating mean flow than random forcing.

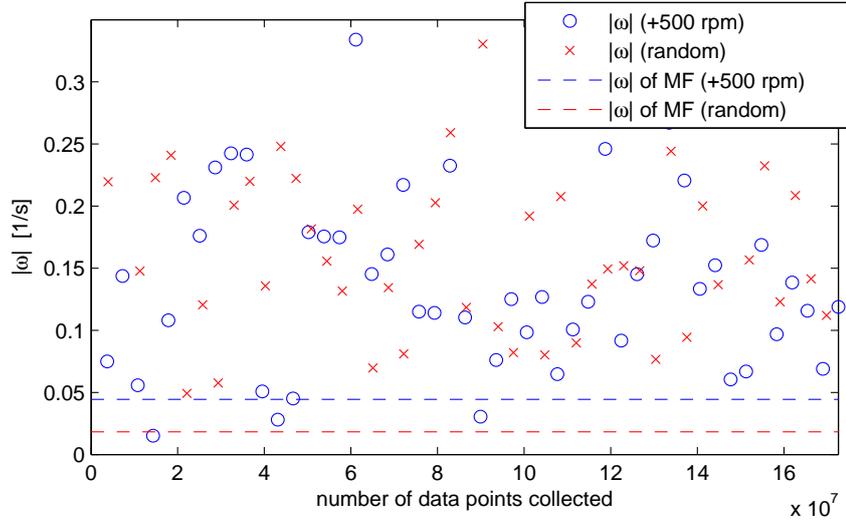


Figure VII.21: Magnitude of the local vorticity computed from the short-time averaged velocity field. The magnitude of the local vorticity,  $\|\omega\|$ , computed from short-time averaged Eulerian velocity field, is several times bigger than the corresponding value calculated from the long-time averaged flow field (see figure VII.20). The value of the long-time averaged flow field is indicated with the blue (forcing with +500 rpm) and red (random forcing) dashed lines. For both, random forcing and forcing with +500 rpm,  $\|\omega\|$  is fluctuating between  $0.05$  and  $0.25 \frac{1}{s}$ . Moreover, no clear trend can be observed.

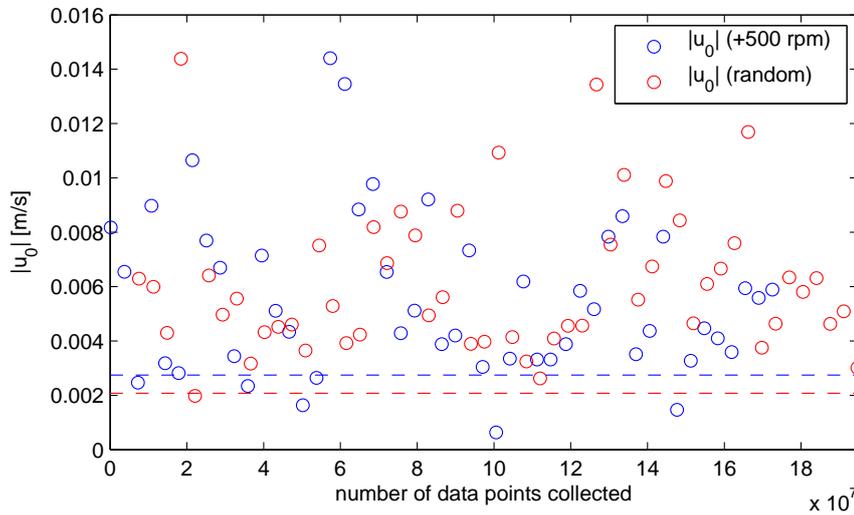


Figure VII.22: Magnitude of  $\mathbf{u}_0$  obtained from the short-time averaged velocity field. The magnitude,  $\|\mathbf{u}_0\|$ , of the velocity computed from short-time averaged (or pseudo-instantaneous) Eulerian velocity field, is several times bigger than the corresponding value calculated from the long-time averaged flow field (see figure VII.18). The value of the long-time averaged flow field is indicated with the blue (forcing with +500 rpm) and red (random forcing) dashed lines. For both forcings,  $\|\mathbf{u}_0\|$  is fluctuating between  $2$  and  $10 \frac{mm}{s}$ . Also, no clear trend can be observed. That agrees with figure VII.21.

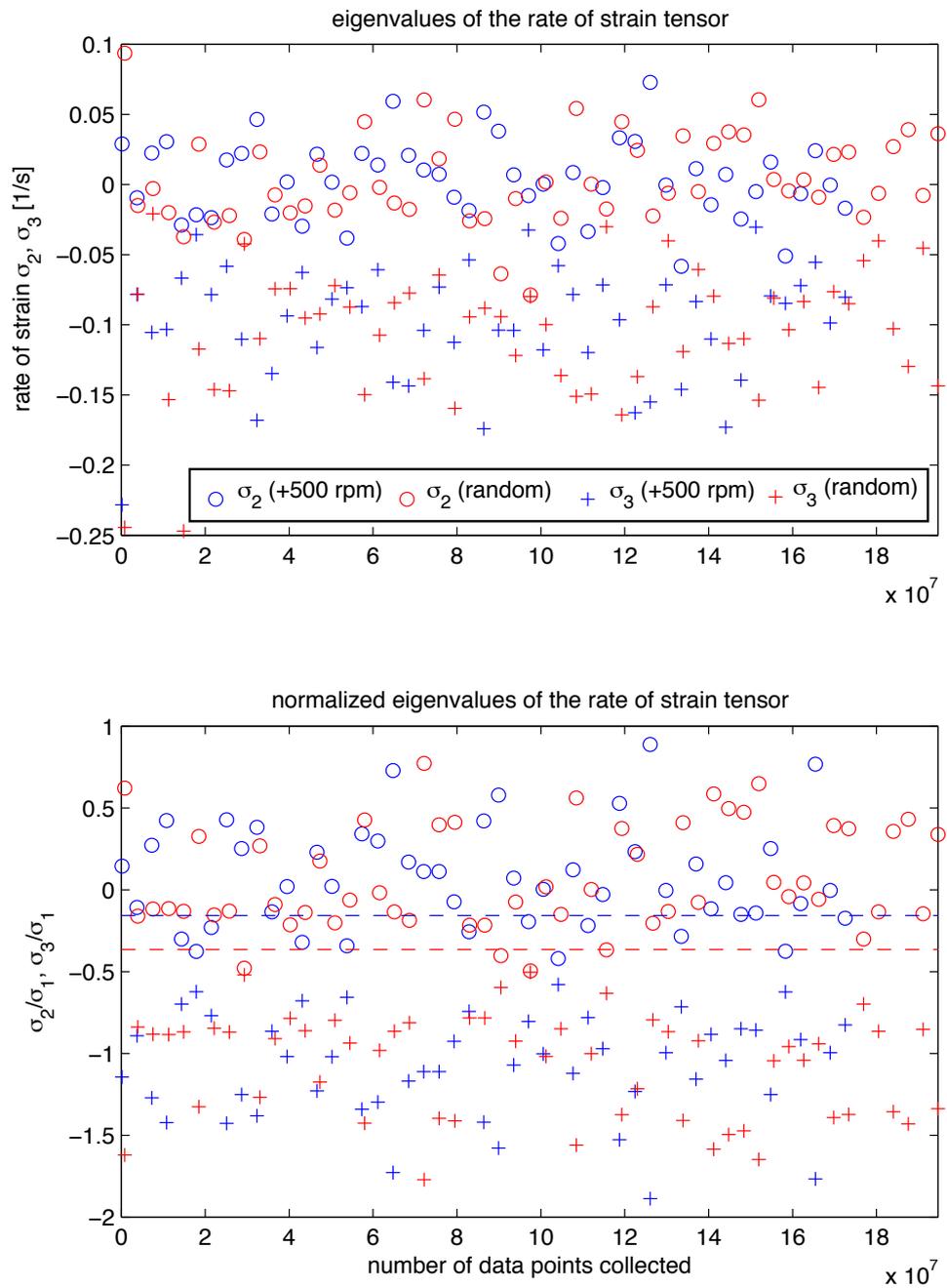


Figure VII.23:  $\sigma_2$  and  $\sigma_3$  of the rate of strain tensor computed from the short-time averaged velocity field. The top figure displays the eigenvalues  $\sigma_2$  and  $\sigma_3$  of the rate of strain tensor computed from short-time averages of the Eulerian velocity field. Surprisingly, it is  $\frac{\sigma_2}{\sigma_1} \gtrsim 0$  for both forcings. The long-time average did not show this (figure VII.19). The figure at the bottom displays  $\sigma_2$  (○) and  $\sigma_3$  (+) normalized by  $\sigma_1$  at random forcing (red) and forcing with +500 rpm (blue). The two dashed lines are the ratio of  $\sigma_2/\sigma_1$  computed for the long-time averaged flow field. Clearly, the values of the short-time averaged mean flow deviate from their long-time average.

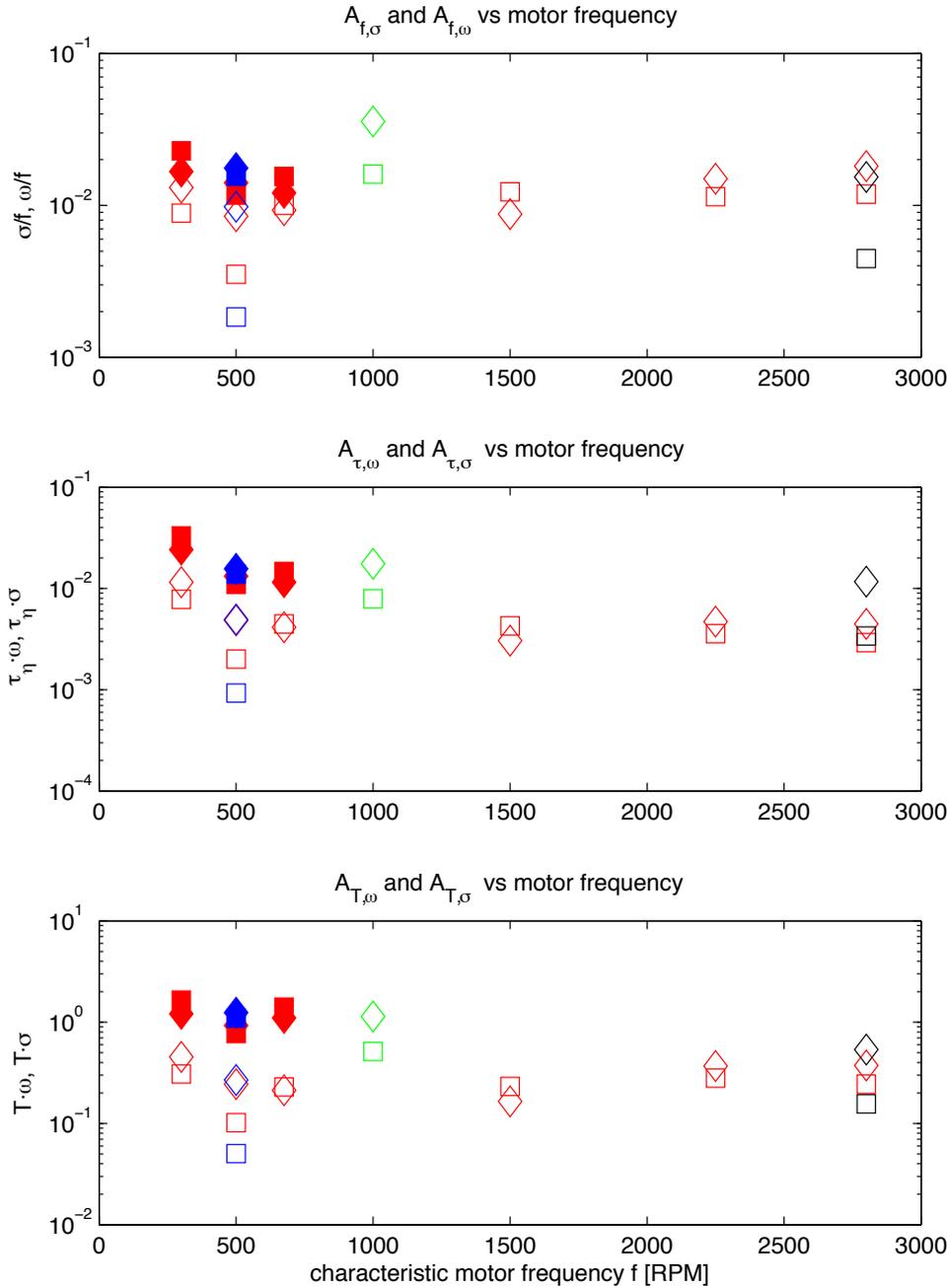


Figure VII.24: Dimensionless parameters  $A_{T,\sigma}$ ,  $A_{T,\omega}$ ,  $A_{f,\sigma}$ ,  $A_{f,\omega}$ ,  $A_{T,\sigma}$ ,  $A_{T,\omega}$  as a function of the motor speed  $f$ . The  $\diamond$  symbol refers to the dimensional parameter based on  $\sigma$  ( $A_{T,\sigma}$ ,  $A_{f,\sigma}$ ,  $A_{T,\sigma}$ ), whereas  $\square$  denotes parameters based on  $\omega$  ( $A_{T,\omega}$ ,  $A_{f,\omega}$ ,  $A_{T,\omega}$ ). The colors denote: red: CW forcing, green: anisotropic forcing, blue: random forcing and black: CCW forcing. The small observation volume runs are marked with filled symbols. The mean flow characterized by rate of strain,  $\sigma$ , and local vorticity,  $\omega$ , changes 5 to 10 times slower than the eddy turn over time. Consequently, it changes a few hundred times slower than the Kolmogorov time scale. Dimensionless numbers based on the Kolmogorov time scale decrease with an increase in motor speed. This supports Kolmogorov's idea of scale separation. The parameters based on propeller speed and rate of strain or vorticity are constant at approximately  $\frac{1}{100}$  but not unity. Therefore, the timescale of the mean flow is not on the order of the propeller speed. Random forcing creates a less rotating flow than the other forcing investigated.

## VII.5 Prediction on the Lagrangian structure functions of position and velocity

The Lagrangian framework enables to obtain auto correlations and structure functions along a trajectory. Moreover, it is possible to investigate models which require precise data in this framework. This section will test predictions by Zybin *et al* in [1] and [28]. Reference [1] was later published in Physical Review Letters [29]. It should be noted, that the prediction on the scaling of the Lagrangian structure functions of position is not included in the P.R.L. publication.

Zybin and his group derived the shape of the PDF of vorticity inside a cylindrical vortex filament<sup>11</sup> in fully developed hydrodynamical turbulence [28]. They restricted the derivation to the inertial range of an incompressible fluid. Moreover, they assumed that the Euler equation (III.7) is sufficient to describe a trajectory in this regime.

A fluid segment or a particle will experience only small changes to its velocity in regions of low vorticity. Whereas, passing a vortex filament leads to fast oscillations of the position of the observed test particle. They claim that the latter contributes most to

$$K_n(\tau) = \langle |\mathbf{u}(t + \tau) - \mathbf{u}(t)|^n \rangle \quad (\text{VII.19})$$

which is the  $n$ -th order Lagrangian structure function of velocity of a time difference  $\tau$ . Assuming that the structure function follows a power law, one can define the (Lagrangian) scaling exponent  $\xi_n^L$ :

$$K_n \propto \tau^{\xi_n^L} \quad (\text{VII.20})$$

Using the shape of the PDF of vorticity in such a filament, they were able to compute the scaling exponents,  $\xi_n^L$ , in the inertial range from a set of differential equations [29]. They compared these to results published by our group [30]. It should be noted, that K41 also predicts the behavior of the scaling exponents. The energy transfer rate has the unit [ $m^2 s^{-3}$ ], thus K41 and dimensional arguments yield

$$K_n \propto (\varepsilon \tau)^{n/2} \quad \Rightarrow \quad \xi_n = \frac{n}{2} \quad (\text{VII.21})$$

We measured the  $K_n$  up to the sixth order in the LEM for clockwise forcing at +500 rpm (bMV). They are displayed in figure VII.26.

Ouellette [26] proposed to define the Lagrangian inertial range as the region where  $K_2$  scales linear with time. The definition becomes clear when focusing on the averaged Lagrangian structure function,  $K_n^{avg}$ , and their corresponding averaged energy transfer rate  $\varepsilon^{avg}$ :

$$K_n^{avg} = \frac{1}{N} \sum_{i=1}^N K_i \quad (\text{VII.22a})$$

$$\varepsilon^{avg} = \frac{1}{N} \sum_{i=1}^N \varepsilon_i \quad (\text{VII.22b})$$

<sup>11</sup>Vortex filaments are regions of high vorticity, they arise as eddies become stretched and advected from bigger eddies (see section III.4).

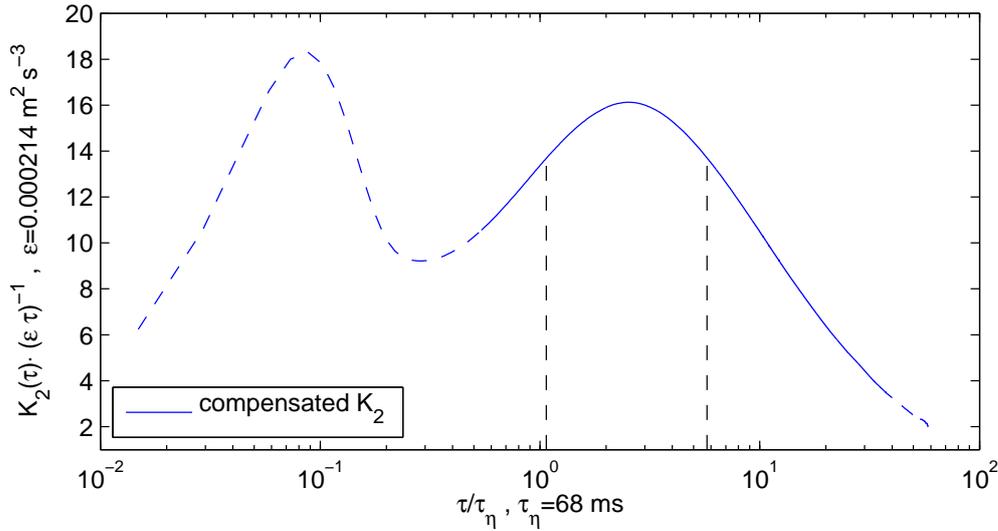


Figure VII.25: Compensated Lagrangian second order structure function of velocity at +500 *rpm* (bMV).  $\frac{K_2(\tau)}{\varepsilon \tau}$  was measured at a propeller speed of +500 *rpm* (bMV). The dashed regions in  $K_2(\tau)$  denote regions of high uncertainty. The inertial range is set to be between the two dashed lines. The latter were chosen such that the value of the compensated  $K_2$  is 85% of the enclosed maximum of  $K_2$ . Thus, the inertial range is between  $1.1 \tau_\eta$  and  $5.7 \tau_\eta$ .

This definition is similar to equations<sup>12</sup> (III.60) and (III.61). Analogous to equation<sup>13</sup> (III.62) it is

$$\left( \frac{1}{N} \sum_{i=1}^N \varepsilon \right)^{n/2} = \frac{1}{N} \sum_{i=1}^N (\varepsilon_i)^{n/2} \quad (\text{VII.23})$$

which holds only for the Lagrangian second order structure function. Although,  $K_2(\tau)$  has yet not been derived from the Navier–Stokes equation, this definition is similar to the definition of the Eulerian inertial range. However, it is common to use  $\frac{\xi_n^L}{\xi_2^L}$ . This enforces  $\xi_2^L = 1$ .

The compensated function  $\frac{K_2(\tau)}{\varepsilon \tau}$  is plotted in figure VII.25. The calculated inertial range is ranging from  $1.1 \tau_\eta$  to  $5.7 \tau_\eta$ . Power law functions were fit with Gnuplot to the inertial range to obtain the  $\xi_n^L$ . The results are shown in figure VII.27 and table VII.5. Our values are in good agreement to a measurement at the same  $R_\lambda \approx 200$  [26, 30]. Although, the predicted values are bigger than our measured scaling exponents [30] indicates that this difference mainly due to the low Reynolds number.

The same methods to obtain the scaling exponents of velocity can be used to predict

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$$^{12} \varepsilon^{avg} = \frac{1}{N} \sum_{i=1}^N \varepsilon \quad \text{and} \quad D_p^{avg} = \frac{1}{N} \sum_{i=1}^N D_i$$

$$^{13} \left( \frac{1}{N} \sum_{i=1}^N \varepsilon \right)^{p/3} = \frac{1}{N} \sum_{i=1}^N (\varepsilon_i)^{p/3}$$

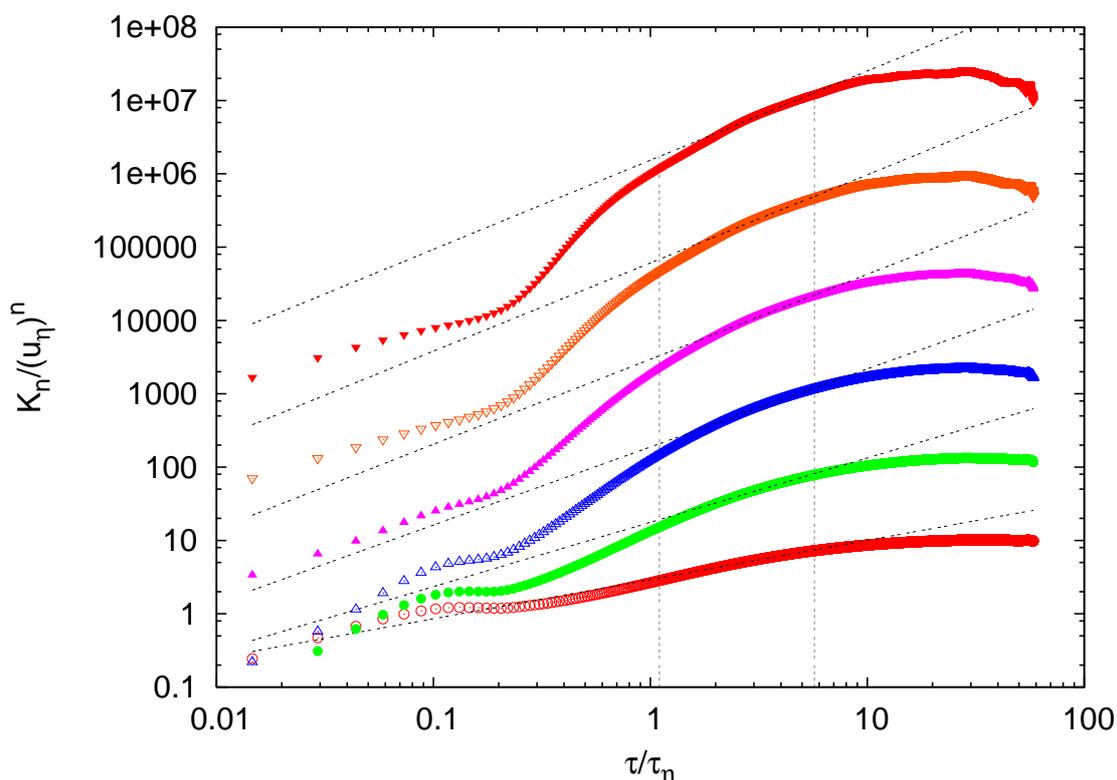


Figure VII.26: Dimensionless Lagrangian  $n$ -order structure function of velocity at +500 rpm (bMV).  $K_n$  was normalized by  $u_\eta^n$  to make the function dimensionless. This scaling introduces only a shift to the  $K_n$ . It does not change the scaling. The order starts from 1 and increases from the bottom to the top. The dashed lines show power laws which were fitted to the inertial range (see figure VII.25). The inertial range is indicated though the two dashed vertical lines.

the behavior of the Lagrangian structure function of position,  $R_n(\tau)$ . The latter is defined as

$$R_n(\tau) = \langle |\mathbf{x}(t + \tau) - \mathbf{x}(t)|^n \rangle \quad (\text{VII.24})$$

They calculated [1] that  $R_n(\tau) \propto \tau^3$  in the inertial range for all  $n > 3$ . The first ten structure functions of position at random forcing and forcing at +500 rpm are shown in figure VII.28 and figure VII.26. In a log-log plot all  $R_n$  with  $n > 3$  should be parallel in the inertial range. Clearly, both figures do not show this behavior. For a further investigation power law functions were fitted to  $R_n(\tau)$  of forcing at +500 rpm in the inertial range. The latter was taken from the Lagrangian structure function,  $K_2$ , as discussed before. The obtained exponents,  $\beta_n^L$ , are shown in figure VII.27 and table VII.4. The measured exponents,  $\beta_n^L$ , seem to be proportional to the order  $n$ . With the help of a linear fit, we determined

$$\beta^L(n) = (0.8984 \pm 0.0088) \cdot n \quad (\text{VII.25})$$

In other words

$$R_n(\tau) \propto \tau^{0.9n} \quad (\text{VII.26})$$

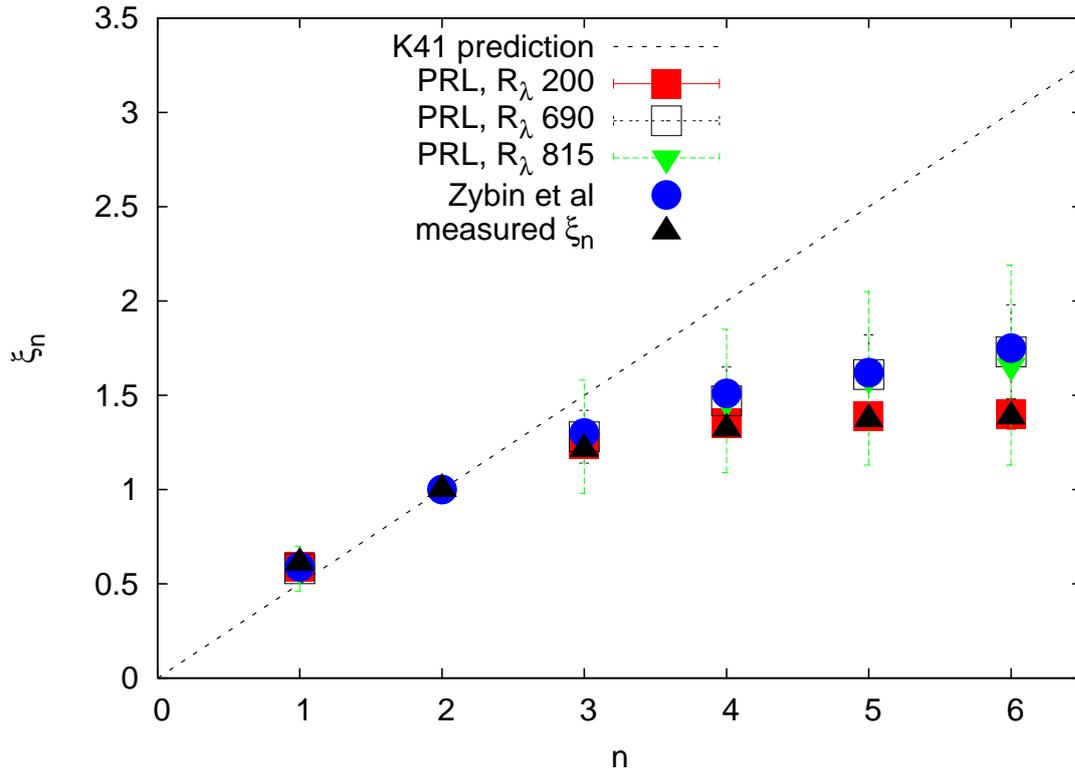


Figure VII.27: Exponents of the Lagrangian structure function of velocity at +500 rpm (bMV). The scaling exponents were obtained in figure VII.26 through power law fits to  $K_n$  in the Lagrangian inertial range. *Zybin et al* denotes the prediction from [29], whereas *PRL* denotes the measurements published [26, 30]. The dashed line show scaling according to K41. Clearly, no K41 scaling was not observed. Our data ( $\blacktriangle$ ) is in very good agreement with the measurement ( $\blacksquare$ ) at the same Reynolds number of  $R_\lambda \approx 200$  [26, 30]. With increasing Reynolds numbers the measurements approach their prediction.

We also plotted the compensated structure function,  $\frac{R_n}{\tau^3} \frac{\tau_\eta^3}{\eta^n}$  for  $n > 3$  in figure VII.29. The compensated  $R_n$  show a plateau region for  $t > 8\tau_\eta$ . However, this region is only a few  $\tau_\eta$  wide and depends on the order,  $n$ , of the structure function. That is in agreement with the results from figure VII.28 and VII.30 discussed before.

Thus, the prediction of Zybin *et al* [1] could not be verified at  $R_\lambda \approx 200$ . However, we can try to relate the linear dependence of  $\beta_n^L$  to its physical meaning. The inner part of  $R_n$ ,  $\mathbf{x}(t + \tau) - \mathbf{x}(t)$  can be expressed with the velocity

$$\mathbf{x}(t + \tau) - \mathbf{x}(t) = \int_t^{t+\tau} \mathbf{u}(t') dt' \quad (\text{VII.27})$$

Inserting (VII.27) into equation (VII.24) yields

$$R_n(\tau) = \langle |\mathbf{x}(t + \tau) - \mathbf{x}(t)|^n \rangle = \left\langle \left| \int_t^{t+\tau} \mathbf{u}(t') dt' \right|^n \right\rangle \quad (\text{VII.28})$$

We can decompose the velocity,  $\mathbf{u}(t')$ , into an average velocity,  $\langle \mathbf{u} \rangle$ , plus fluctuations in the velocity,  $\mathbf{u}'(t')$ . The integral (VII.27) transforms then into

$$\int_t^{t+\tau} \mathbf{u}(t') dt' = \langle \mathbf{u} \rangle \tau + \int_t^{t+\tau} \mathbf{u}'(t') dt' \quad (\text{VII.29})$$

If  $\int_t^{t+\tau} \mathbf{u}'(t') dt'$  is small compared to  $\langle \mathbf{u} \rangle \tau$ , we get

$$R_n(\tau) = \langle |\mathbf{u} \tau|^n \rangle \quad (\text{VII.30})$$

In other words,

$$R_n \propto \tau^n \quad (\text{VII.31})$$

which is close to our observation  $R_n \propto \tau^{0.9n}$ . The deviation is probably mainly due to the negligence of  $\int_t^{t+\tau} \mathbf{u}'(t') dt'$ . Moreover, measurement errors introduce a further bias. Yet, this first approximation yields a result comparable to the observation.

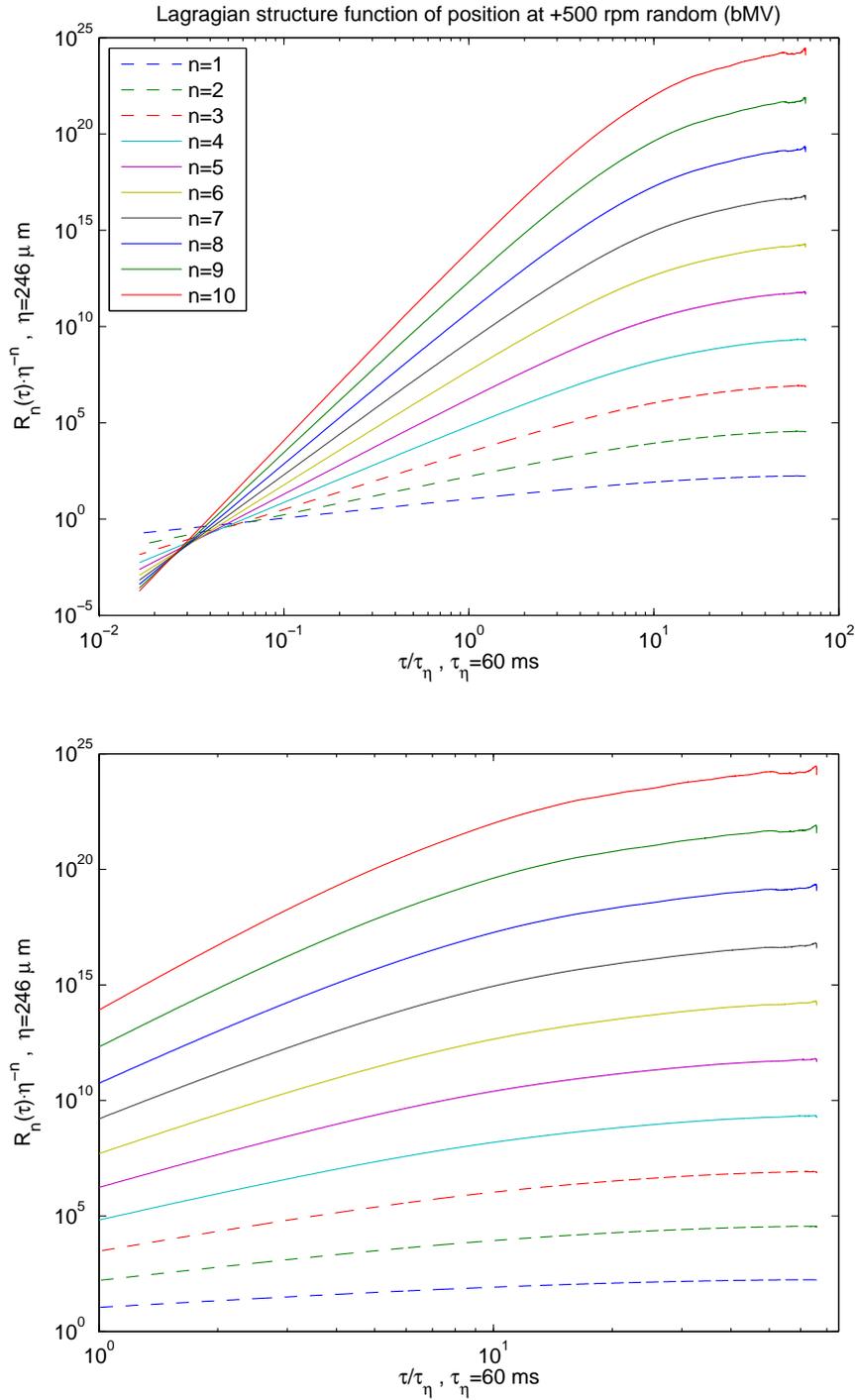


Figure VII.28: Dimensionless Lagrangian  $n$ -order structure function  $R_n/\eta^n$  of position. The top figure shows the Lagrangian structure functions of position for the random forcing (bMV). The bottom figure is an enlargement of the probable inertial range, the legend did not change. For the first three  $R_n/\eta^n$  (dashed curves) no prediction was made. According to [1] all  $R_n/\eta^n$  with  $n > 3$  should scale with  $\tau^3$ . Therefore, the  $R_n/\eta^n$  should be approximately parallel in the inertial range. Our data allows us to cover a possible inertial range between  $1\ \tau_\eta$  and approximately  $40\ \tau_\eta$ . Obviously, the  $R_{n>3}$  are not parallel in this range. Moreover, using the inertial range found in figure VII.25, the scaling exponents,  $\beta_n^L$  are increasing with the order,  $n$ , of  $R_n$ .

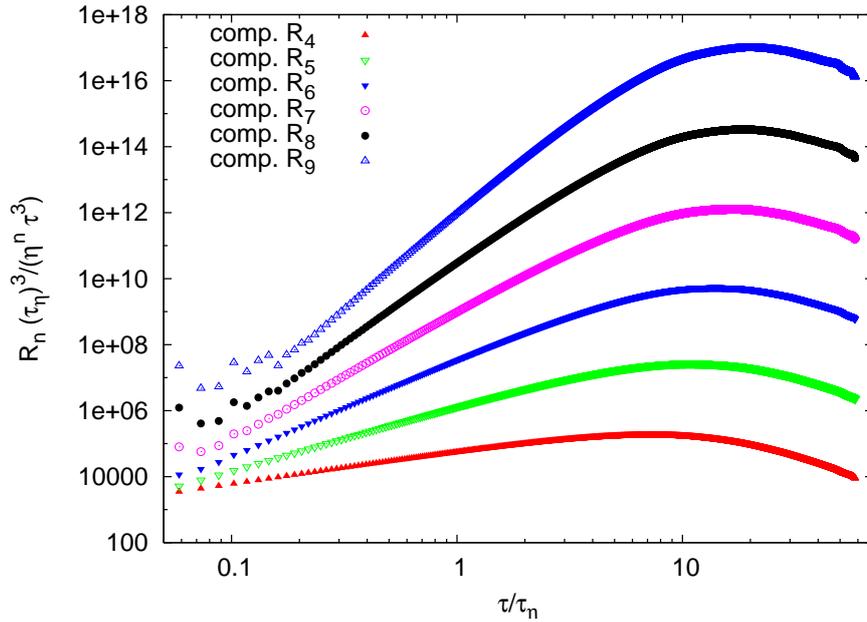


Figure VII.29: Compensated Lagrangian  $n$ -order structure function of position. Clockwise forcing at  $+500 \text{ rpm}$  (bMV) was used to obtain the compensated dimensionless Lagrangian structure function of position  $\frac{R_n}{\tau^3} \frac{\tau_\eta^3}{\eta^n}$  for  $n > 3$ . A plateau region can be observed for  $t > 8\tau_\eta$ . However, this region is only a few  $\tau_\eta$  wide, and the location of the peak depends on the order,  $n$ , of the structure function.

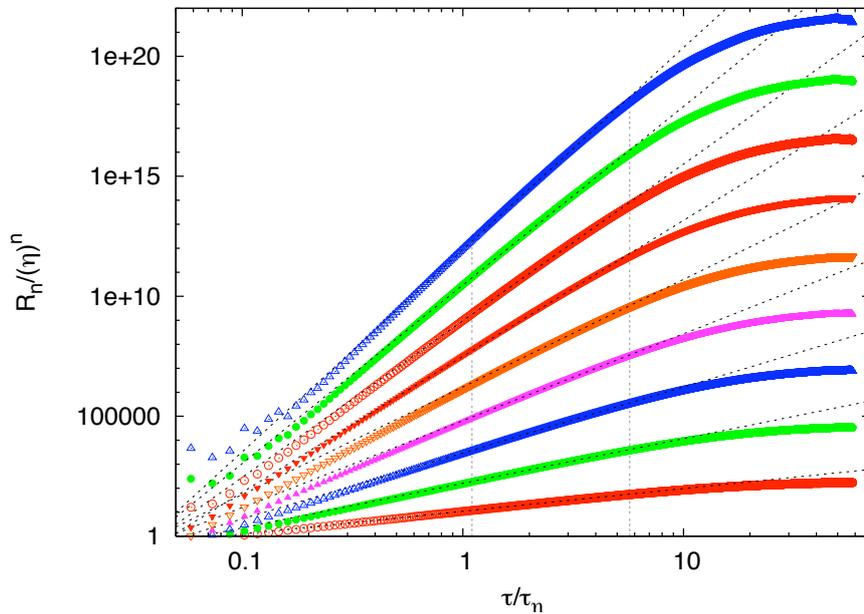


Figure VII.30: Dimensionless Lagrangian  $n$ -order structure function of position. Clockwise forcing at  $+500 \text{ rpm}$  (bMV) was used to obtain the dimensionless Lagrangian structure function of position  $R_n / \eta^n$ . The order,  $n$ , is starting from 1 and increasing from the bottom to the top. The black dashed lines show power laws which were fitted to the inertial range (as obtained in figure VII.25).

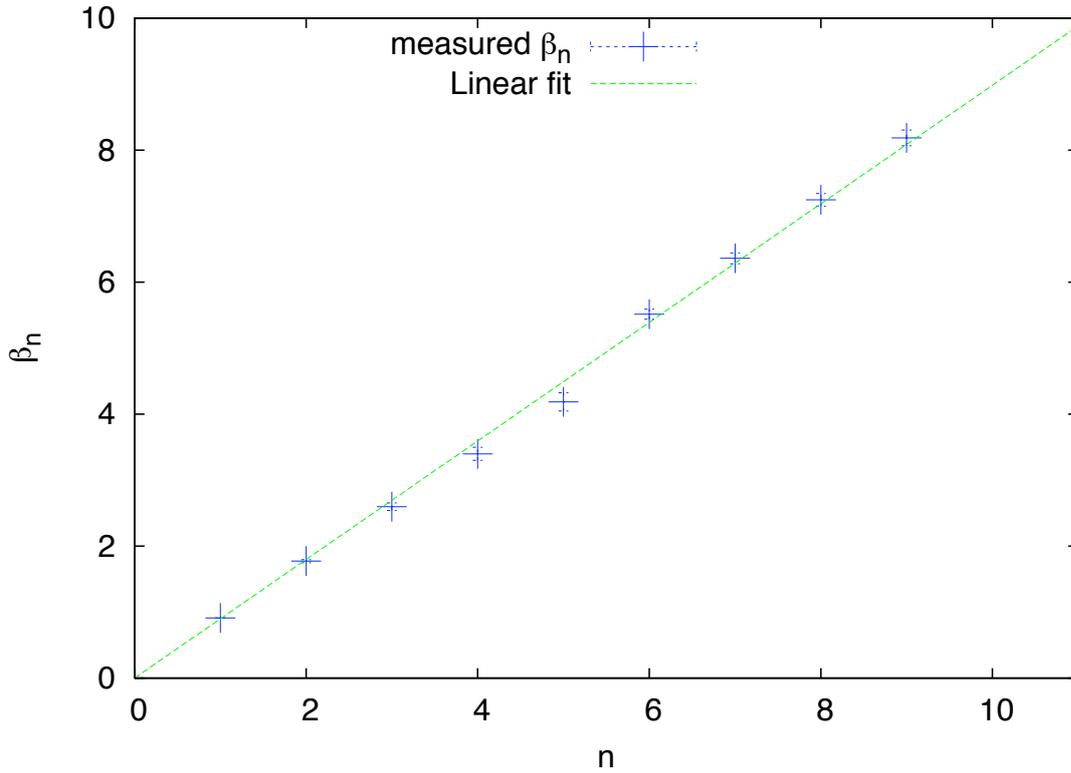


Figure VII.31: Exponents of the Lagrangian structure function of position. Power laws were fitted to the Lagrangian structure function of position with forcing at +500 rpm (figure VII.30) to the inertial range (see figure VII.25). According to [1], the scaling exponents obtained,  $\beta_n^L$ , should be 3 for order four and higher. Clearly, this was not observed. On the contrary, the exponents,  $\beta_n^L$ , increase linear with  $n$  as indicated by the linear fit,  $\beta^L(n) = (0.8984 \pm 0.0088) \cdot n$ . In other words,  $R_n(\tau) \propto \tau^{0.9n}$ .

Table VII.4: Measured scaling exponents  $\beta_n^L$  the Lagrangian structure function of position  $R_n$ . The measured  $\beta_n^L$  were obtained at  $R_\lambda \approx 200$  at forcing with +500 rpm (bMV).

Order $n$	1	2	3	4	5	6	7	8	9
$\beta_n^L$	0.91	1.77	2.60	3.40	4.19	5.51	6.36	7.25	8.2

Table VII.5: Measured and predicted Lagrangian scaling exponents  $\frac{\xi_n^L}{\xi_2^L}$ . The measured  $\xi_n^L$  were obtained at  $R_\lambda \approx 200$  at forcing with +500 rpm (bMV).

Order $n$	$\frac{\xi_n^L}{\xi_2^L}$	value from [29]
1	0.61	0.59
2	1	1
3	1.21	1.30
4	1.32	1.51
5	1.37	1.65
6	1.38	1.75

## VIII Conclusion and Outlook

The Lagrangian Exploration Module was characterized with Particle Tracking Velocimetry at different propeller speeds and different forcings. It was demonstrated that the LEM produced nearly isotropic and homogeneous turbulence in a volume of the size of the integral length scale. The turbulence was characterized by values of  $R_\lambda$  up to about 350. Higher turbulence levels under similar experimental conditions are most likely achievable by increasing the speed of the propeller, or by using a different propeller design. The mean flow was shown to be on the order of a few  $\frac{mm}{s}$ , and the fluctuations in velocity were found to be several times stronger than the underlying mean-flow. Thus, the LEM is a good tool for Lagrangian experiments.

It is possible to change the properties of the flow by adjusting the speed of each propeller individually. We investigated the difference in the turbulence and the mean flow field by driving the flow in two different ways. In the first, the propeller speed was held constant and in the second, the propeller speed changed randomly over time. We ensured that the energy rate injected was the same in both experiments. Only small variations in the turbulence and the mean flow were observed between the data obtained with these two different forcings. This recalls the idea of Kolmogorov, which is that the statistics at scales much smaller than the scales at which the turbulence is created are independent of the way the turbulence was created.

The predictions by Zybin *et al* [1, 28, 29] were compared to results obtained in the LEM at  $R_\lambda \approx 200$ . Good agreement was found for the scaling exponents,  $\xi_n^L$ , of the Lagrangian  $n$ -th order structure functions of velocity,  $K_n(\tau)$ . The scaling exponents obtained in the LEM were also in very good agreement to those obtained in a French-Washing machine at the same turbulence level [26, 30].

Furthermore, the scaling exponents,  $\beta_n^L$ , of the Lagrangian  $n$ -th order structure function of position,  $R_n(\tau)$ , were measured at  $R_\lambda \approx 200$ . In contrast to [1],  $\beta_n^L$  did not appear to be 3 for  $n > 3$ . Moreover, we observed  $\beta_n^L = (0.898 \pm 0.009) \cdot n$  for  $n < 10$ . Therefore, this prediction is not supported by our experiments at the turbulence level investigated.

A possible future project for the Lagrangian Exploration Module is a detailed study on the question of how to generate turbulence. After automatizing the measuring and analysis process, one could use genetic algorithms to explore the space of possible forcings in much shorter time than with our present analysis methods. The insight gained through this research could also improve the design of actively-driven grids in wind-tunnels.

The large number of windows on the LEM enables us to use several measurement techniques at the same time, thus allowing more stringent tests and comparisons of new or existing instruments. For example, new physical insight can be gained

through the simultaneous observation of particles of different sizes or types, and from the combination of Eulerian with Lagrangian measurement techniques.

A further possible project is to use pressurized sulfur-hexafluoride ( $SF_6$ ) as a working fluid in the LEM. The Göttingen U-Boot pressure vessel provides  $SF_6$  at pressures up to 15 *bar*.  $SF_6$  has kinematic viscosity of  $1.5 \cdot 10^{-7} m^2 s^{-1}$  at this pressure. Since the maximum turbulence level observable is limited mainly by the Kolmogorov time scale and the frame rate of the cameras,  $SF_6$  would allow us to reach higher turbulence levels.

In addition, it is planned to use the LEM to investigate the behavior of copepods in a turbulent environment. These small scrimp-like animals are an important food source for fish larvae.

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