A METHOD OF SOLVING THE DIRICHLET PROBLEM
FOR THE SECOND-ORDER ELLIPTIC EQUATION IN A POLYGONAL
DOMAIN ON AN ADAPTIVELY REFINED MESH SEQUENCE

E.S. Nikolaev and O.V. Shishkina

A version of the finite element method with piecewise-linear basis functions
which generates a sequence of adaptively refined triangular grids is constructed
for finding, in a polygonal domain, a generalized solution of the Dirichlet
problem for the second-order linear elliptic equation with a symmetric operator.
Two strategies of mesh refinement are considered which are based on the
use of a posteriori estimation of the error, namely, the correction indicator,
and a new procedure of reconstruction of the domain triangulation is pro-
posed. For problems whose solutions possess an exponential-type singularity
the constructed adaptive method makes it possible to decrease dozens of
times the number of nodes as well as the time needed for computations as
compared to the finite element method which uses the strategy of uniform mesh
refinement.

1. INTRODUCTION

The finite element methods (FEM) for solving boundary value problems for differential equations
which use the adaptive strategies of mesh refinement and local variation of the order of elements (h- and
p-versions) have been actively developed since the end of the 1970s. These methods are based either on
the direct use of local a posteriori estimates of the errors, for which purpose a mathematical apparatus was
developed in [1-6], or on the computation of correction indicator, i.e., a quantity estimating the degree of
variation of the characteristic chosen for an approximate solution upon adding hierarchically one or several
degrees of freedom [7-11]. The present-day state of investigations concerning the technique of constructing
a posteriori estimates is reflected in reviews [12-15] and the differences between these classes of adaptive
methods are discussed in detail in [8].

In addition to the indicated methods, we want to point out methods generating adaptive distribution of
the grid notes for the case where their number is defined (the r-version of FEM) [16-18] as well as methods
based on the rearrangement of the given initial grid by means of a direct minimization of the functional
relative to the coordinates of the grid nodes [19-21] for problems which admit a variational formulation.

In this work, for an approximate solution of the Dirichlet problem for a linear elliptic second-order
equation defined in a polygonal domain, we construct a version of the finite element method with piecewise-
linear basis functions which generates a sequence of triangular grids refined and adapted to the solution.
The proposed strategy of mesh refinement is based on the use of a correction indicator, i.e., a posteriori
estimate of the variation in C-norm of an approximate solution upon the addition to the grid of a test
node and on completing hierarchically (with the preservation of the class) the set of basis functions by the
addition of a new function which is associated with this node. New procedures of the rearrangement of
the triangulation of the domain for the process of adaptive mesh refinement have been constructed. Numerical
experiments have been performed for estimating the efficiency of the constructed method with the use of an

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example of a singularly perturbed problem whose solution has an exponential-type singularity. This version
is compared to the standard variant of the finite-element method which uses the strategy of uniform mesh
refinement.

2. MINIMIZATION PROBLEM AND ITS GRID ANALOG

We have to solve a problem of functional minimization, namely, in the polygonal domain \( \overline{G} = G + \partial G \)
with boundary \( \partial G \) we have to find a function \( u^\ast(x) \) such that

\[
\begin{align*}
  u^\ast = \arg \min_{u \in H} J(u), \quad u^\ast \in H = \{ u \in W^1_2(G), \, u(x) = u_0(x), \, x \in \partial G \},
\end{align*}
\]

where \( J(u) \), the bilinear \( a(u,v) \) and linear \( b(v) \) forms are defined as

\[
  J(u) = \frac{1}{2} a(u,u) - b(u), \quad a(u,v) = \int_G \left( \sum_{\alpha,\beta=1}^2 k_{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial x_\beta} + q_{\alpha\beta} u v \right) dx, \quad b(v) = \int_G f v \, dx.
\]

The following conditions are assumed to be fulfilled:

1. the matrix \( k = \{ k_{\alpha\beta}(x) \}_{\alpha,\beta=1}^2 \) is symmetric and uniformly, with respect to \( x \in \overline{G} \), positive definite,
   \( q = q(x) \) is a nonnegative function,

2. the function \( u_0 \) is continuous on \( \partial G \) and admits of a continuation into \( W^1_2(G) \), i.e., there exists a
   function \( v \in W^1_2(G) \) such that its trace on \( \partial G \) coincides with \( u_0 \).

By virtue of assumptions made for \( k \) and \( q \), the form \( a(\cdot, \cdot) \) is symmetric and coercive on the functions
which vanish on \( \partial G \).

If \( k_{\alpha\beta}, q \in L_\infty(G) \), and \( f \in L_2(G) \), then a solution of problem (1), (2) exists, is unique, and is a
generalized solution of the Dirichlet problem for the second-order elliptic equation

\[
  - \sum_{\alpha,\beta=1}^2 \frac{\partial}{\partial x_\alpha} \left( k_{\alpha\beta} \frac{\partial u}{\partial x_\beta} \right) + qu = f(x), \quad x \in G, \quad u(x) = u_0(x), \quad x \in \partial G.
\]

Now if \( k_{\alpha\beta} \in C^1(G) \) and \( q, f \in C(G) \), but \( u^\ast \in C^2(G) \cap C(\overline{G}) \), then \( u^\ast \) is the classical solution of problem (3).

Grid problem. The method for solving problem (1), (2) proposed below is based on the finite element
approach to the approximate determination of the point of minimum of the functional with the use of a
sequence of adaptive mesh refinement.

Suppose that we have constructed in a domain a triangulation \( \overline{G} = \bigcup_{i=1}^{n_\omega} T^i \) containing \( n_\omega \) triangles \( T^i \)
and \( n_v \) vertices belonging to the set of nodes \( \omega = \omega + \partial \omega \), where

\[
  \omega = \{ x_i \in G, \, 1 \leq i \leq n \}, \quad \partial \omega = \{ x_i \in \partial G, \, 1 \leq i \leq m \}, \quad n_v = n + m.
\]

The grid is defined by the coordinates of nodes and the graph of the triangulation which indicates the nodes
of the grid to which every node is connected. The edges of the graph are associated with the sides of the
triangles and the vertices with the corresponding nodes of the grid. We shall consider only conformal grids.
The grid is said to be conformal if the intersection of any two triangles from the triangulation is either a
line intercept connecting the vertices of these triangles or the common vertex, or an empty set.

Let \( \{ \phi_j \}_{j=1}^n \) and \( \{ \psi_j \}_{j=1}^m \) be collections of piecewise-linear continuous basis functions associated with the
nodes from \( \omega \) and \( \partial \omega \). The functions \( \phi_j(x) \) and \( \psi_j(x) \) are equal to unity at the nodes with which they are
associated and to zero at other nodes. We denote by \( \tilde{H}_n = \{ w = \sum_{j=1}^n w_j \phi_j \} \) the set of piecewise-linear
continuous on \( \overline{G} \) functions which vanish on \( \partial G \).
The finite element minimization problem corresponding to (1) is formulated as follows: in the domain \( \mathcal{C} \) we have to find a function \( u(x) \) such that

\[
    u = \arg \min_{v \in H_{n,m}} J(v), \quad H_{n,m} = \left\{ u = \bar{u} + g, \quad \bar{u} \in \breve{H}, \quad g = \sum_{j=1}^{m} u_j(x_j) \psi_j \right\}.
\]

It is known that the function \( u(x) \) can be found from Galerkin's condition

\[
    a(u,v) = b(v) \quad \forall v \in \breve{H}.
\]

Since the basis in the finite-dimensional linear space \( \breve{H} \) is formed by the functions \( \varphi_i \), this condition can be written as a system of linear algebraic equations

\[
    A w = b,
\]

\[
    A = \{ a_{ij} \}_{i,j=1}^{n}, \quad w = (w_1, \ldots, w_n)^T, \quad b = (b_1, \ldots, b_n)^T,
\]

\[
    a_{ij} = a(\varphi_i, \varphi_j), \quad b_i = b(\varphi_i) - a(\varphi_i, \varphi_i), \quad 1 \leq i, j \leq n.
\]

By virtue of the definition of \( \varphi_i(x) \), the symmetry and coerciveness of the form \( a(\cdot, \cdot) \) on the functions from \( \breve{H} \), as well as the relation

\[
    v^T A w = a(w, n), \quad w = \sum_{j=1}^{n} w_j \varphi_j \in \breve{H}, \quad v = \sum_{j=1}^{n} v_j \varphi_j \in \breve{H},
\]

which is valid for any vectors \( w = (w_1, \ldots, w_n)^T, v = (v_1, \ldots, v_n)^T \), the matrix \( A \) is rarefied symmetric and positive definite. The value of the functional on the function \( u \in H_{n,m} \) satisfying (4) can be found from the relation

\[
    J(u) = -0.5 \sum_{j=1}^{n} b_j w_j + J(g).
\]

The exactness of the approximate solution of problem (1), (2) depends on the choice of the set \( \mathcal{W} \) and the graph of triangulation, but it is impossible to define a priori the grid on which the required exactness can be attained. The use of uniformly refined grids for the classes of problems indicated in Introduction may lead to considerable labor intensity in computations. The efficient technique of solving this problem is the iterative method of constructing approximate solutions on a sequence of adaptively refined meshes proposed below. In this work, we understand the grid refinement as the addition of new nodes, arranged on the sides of some triangles, to the set \( \mathcal{W} \) as well as new edges to the graph of triangulation which are needed for the preservation of conformity of the new grid.

The strategies of adaptive mesh refinement proposed in Sec. 3 define the rules of choice of triangles and line segments of the boundary of the triangulation for localizing the new nodes of the grid. It is assumed that the added nodes will be located only at the middle of some sides of the chosen triangles and line segments of the boundary. The procedures of mesh refinement presented in Sec. 4 define the algorithms of generation of the new set of nodes by complementing the old set as well as by constructing a new triangulation with the use of the graph of the old triangulation. The iteration method of solving problem (1), (2) is described in Sec. 5.

3. STRATEGIES OF ADAPTIVE MESH REFINEMENT

The proposed strategies are based on the use of two nonnegative piecewise-constant functions \( I \) and \( R \). The first of them is a correction indicator whose value \( I' \) on the triangle \( T' \) estimates the variation of the
chosen characteristic of the approximate solution obtained upon the arrangement on \( T^i \) of the test node and introduction into consideration of a piecewise-linear function, associated with it, with a support concentrated on this element (just as in the hierarchical \( h \)-version of FEM). The second function estimates the precision of approximation of the function \( u_b(x) \) by the piecewise-linear function \( g(x) \) on each line segment \( \partial G_i \) of the boundary (the interval between the adjacent nodes on the boundary). The value of \( R^i \) on the line segment \( \partial G_i \) is defined as

\[
R^i = \frac{1}{\text{mes}(\partial G_i)} \int_{\partial G_i} (u_b - g)^2 \, dx. \tag{7}
\]

**Strategy of mesh refinement.** The rule of choice of triangles is the following. \( T^i \) is included into the list \( \text{List}_n \) of the chosen triangles if the condition

\[
I^i \geq I_{\text{bar}}, \quad I_{\text{bar}} = \frac{1}{2} \max_{1 \leq i \leq n} I^i + \frac{1}{\mu} \sum_{i=1}^{n} I^i,
\]

where \( \theta \in [-1, 1] \) is the parameter controlling the degree of mesh refinement, is fulfilled. For \( \theta = -1 \) all triangles are taken and for \( \theta = 1 \) only those which have the maximal value of the correction indicator.

The rule of choice of the line segments of the boundary is defined as follows: \( \partial G_i \) is included into the list \( \text{List}_a \) of the chosen segments if the condition

\[
R^i \geq \max(\mu R_{\text{max}}, \varepsilon_u^2), \quad R_{\text{max}} = \max_{1 \leq i \leq m} R^i,
\]

where \( \mu \in [0, 1] \) is the parameter used for controlling the degree of mesh refinement and \( \varepsilon_u \) is the required exactness of the problem solution, is fulfilled.

In this article, we consider two types of correction indicator, \( I_1 \) and \( I_2 \). The corresponding strategies of mesh refinement are denoted by \( S_1 \) and \( S_2 \). The indicator of the first type estimates the decrease of the functional value and that of the second type estimates the change of the solution at the added node. When calculating the indicators, we shall distinguish between the internal and boundary triangles, i.e., triangles one or two sides of which (called boundary sides) belong to \( \partial G \). For each triangle \( T^i \) the values of the indicators \( I^i_1 \) and \( I^i_2 \) are calculated upon the addition to the set \( \{\psi_j\}^{n}_{j=1} \) of a function \( \psi^i(x) \) associated with the center of this triangle. For boundary triangles similar quantities are additionally calculated when the function \( \psi^i(x) \) associated with the middle of the boundary side is added to the set \( \{\psi_j\}^{m}_{j=1} \).

### 3.1. Correction Indicators for all Triangles

At the intersection of the medians of the triangle \( T^i \) we place a test node \( x^i \) whose connection to the vertices of the triangle defines the new triangulation of the domain. We complement the set \( \{\varphi_j\}^{n}_{j=1} \) by a piecewise-linear basis function \( \varphi^i(x) \) associated with the node \( x^i \) and define the set of functions

\[
H_{n+1}^i = \left\{ w^i = \sum_{j=1}^{n} w_j \varphi_j + w_{n+1} \varphi^i \right\}, \quad H_{n+1,m}^i = \left\{ u = w^i + g, \; w^i \in H_{n+1}^i, \; g = \sum_{j=1}^{m} u_b(x_j) \psi_j \right\}.
\]

Let \( u \in H_{n,m}^i \) be an approximate solution of problem (1) on the grid with \( n \) internal and \( m \) boundary nodes, i.e., a function satisfying condition (4). Let \( u^i \in H_{n+1,m}^i \) be an approximate solution of problem (1) on the test grid, i.e., a function satisfying the condition

\[
a(u^i, v^i) = b(v^i) \quad \forall v^i \in H_{n+1}^i. \tag{8}
\]

We denote by \( J_1^i \) and \( J_2^i \) the quantities

\[
J_1^i = J(u) - J(u^i), \quad J_2^i = \max_{x \in T^i} |u(x) - u^i(x)|,
\]
which characterize the variation of the functional $J(v)$ and of the solution in the triangle $T^i$ upon the transition from the minimization of the functional on the set $H_{n,m}$ to its minimization on the set $H_{n+1,m}^i$.

**Theorem 1.** The representations

$$J_1^i = 0.5w_{n+1}^i [b(\varphi^i) - a(u, \varphi^i)],$$

$$J_2^i = \max_{x \in T^i} |y'(x) - \varphi'(x)|,$$

$$u - u^i = w_{n+1}^i (y^i - \varphi^i),$$

$$w_{n+1}^i = \frac{b(\varphi^i) - a(u, \varphi^i)}{a(\psi^i, \varphi^i) - a(y^i, \varphi^i)}$$

(9)

(10)

hold true, where the function $y^i \in \tilde{V}_n$ is defined by the condition

$$a(y^i, v) = a(\varphi^i, v) \quad \forall v \in \tilde{H}_n.$$  

(11)

**Proof.** We shall show, for the beginning, that the relations

$$a(z^i, \varphi^i) = a(u, \varphi^i) - b(\varphi^i),$$

$$a(z^i, v) = 0 \quad \forall v \in \tilde{H}_n$$

(12)

hold for $z^i = u - u^i$. Indeed, using (8) with $v^i = \varphi^i$, we obtain the first relation. Since $\tilde{H}_n \subset \tilde{H}_{n+1}^i$, the second relation follows immediately from (4) and (8).

Let us prove the validity of relations (10). For $v = z^i - w_{n+1}^i (y^i - \varphi^i) \in \tilde{H}_n$, with due account of (11) and the second relation from (12), we obtain $a(v, v) = a(z^i, v) - w_{n+1}^i a(y^i - \varphi^i, v) = 0$. Since the quadratic form is positive on the nonzero functions from $\tilde{H}_n$, this relation holds if and only if $v = 0$. Hence follows the first formula from (10). Substituting the representation obtained for $z^i$ into the first of relations (12), we get formula (10) for $w_{n+1}^i$. Let us show that the denominator of this formula is nonzero. Using condition (11) with $v = y^i$, we obtain $a(\varphi^i - y^i, \varphi^i) = a(\varphi^i - y^i, \varphi^i - y^i)$. Since the quadratic form is positive on the nonzero functions from $\tilde{H}_{n+1}^i$ and $\varphi^i \notin \tilde{H}_n$, it follows that $a(\varphi^i, \varphi^i) - a(y^i, \varphi^i) > 0$.

Using the symmetry of the form $a(u, v)$ and relation (8) with $v^i = z^i \in \tilde{H}_{n+1}^i$, we have

$$2J_1^i = a(u, u) - a(u, v^i) - 2b(z^i) = a(z^i, v) - w_{n+1}^i a(z^i, \varphi^i),$$

where $v = z^i + w_{n+1}^i \varphi^i \in \tilde{H}_n$. From this relation and from (12) follows formula (9) for $J_1^i$. The theorem is proved.

Theoretically, the quantities $J_1^i$ and $J_2^i$ can be used as correction indicators. In this case, the main objective will be the solution of problem (11) for finding the function $y^i$. This objective reduces to solving system (5) in a partial statement with a special rarefied right-hand side, and for large $n$, the solution of all these problems will require considerable computations. Therefore it is desirable to choose, as correction indicators, quantities which are close to $J_1^i$ and $J_2^i$ and which are sufficiently easy to calculate. We define the values of the correction indicators $I_1^i$ and $I_2^i$ on the triangle $T^i$ as follows:

$$I_1^i = \frac{[b(\varphi^i) - a(u, \varphi^i)]^2}{2a(\varphi^i, \varphi^i)},$$

$$I_2^i = \frac{[b(\varphi^i) - a(u, \varphi^i)]}{a(\varphi^i, \varphi^i)}.$$

(13)

This choice of the correction indicators will be justified in Sec. 3.3.

Note that on each of these three triangles generated by the location of the test node in $T^i$ the function $\varphi^i$ is represented as a linear combination of basis functions associated with the vertices of the triangle $T^i$. Therefore it is not necessary to find any additional integrals for calculating the introduced correction indicators.
3.2. Correction Indicators for Offboundary Triangles

We denote by $\hat{I}_1$ and $\hat{I}_2$ the values of correction indicators for the offboundary triangle $T^i$ calculated from relations (13). Additionally we shall find similar quantities $\bar{I}_1$ and $\bar{I}_2$ adding a new node to the middle of the boundary side. These quantities will characterize the change of the functional and the solution when we replace, on this side, the linear interpolation of the function $u_0$, with the use of the function $g$, by a more exact piecewise-linear interpolation by the function $g^i$. We define the values of correction indicators for the offboundary triangle as follows:

$$
\hat{I}_1 = \max(\hat{I}_1, \bar{I}_1), \quad \hat{I}_2 = \max(\hat{I}_2, \bar{I}_2).
$$

If two sides of the triangle $T^i$ belong to the boundary $\partial G$, then the quantities $\hat{I}_1$ and $\hat{I}_2$ are calculated for each of these sides.

Thus, we place a test node $x^i$ onto the middle of the boundary side with endpoints $x_l$ and $x_k$ and connect this node with the vertex of the triangle lying opposite to this side. As a result, we obtain a new triangulation of the domain. We complement the set $\{\psi_j\}_{j=1}^m$ by a piecewise-linear basis function $\psi^i$ associated with the node $x^i$ and define the sets of functions

$$
\hat{H}^i_n = \{ w^i = \sum_{j=1}^m w_j^i \psi_j \}, \quad H^{i+1}_{n+1} = \{ u = w^i + g^i, \; w^i \in \hat{H}^i_n, \; g^i = \sum_{j=1}^m u_0(x_j)\psi_j + g^i_{m+1}\psi^i \}.
$$

Since the relation $g^i(x) = u(x)$ must hold for $x = x^i$, it follows that

$$
g^i_{m+1} = u_0(x^i) - \frac{u_0(x_l) + u_0(x_k)}{2},
$$

(14)

Let $u \in H^i_{n,m}$ be an approximate solution of problem (1) on a grid with $n$ internal and $m$ boundary nodes, i.e., a function satisfying condition (4). Let $w^i \in H^i_{n,m+1}$ be an approximate solution of problem (1) on a test grid, i.e., a function satisfying the condition

$$
a(u^i, v) = h(u) \quad \forall v \in \hat{H}^i_n.
$$

As before, we denote $J^i_1 = J(u) - J(u^i)$ and $J^i_2 = \max_{x \in T^i} |u - u^i(x)|$.

**Theorem 2.** The representations

$$
J^i_1 = g^i_{m+1} \left[ h(\psi^i) - a(u, \psi^i) \right] - 0.5g^i_{m+1} \left[ a(\psi^i, \psi^i) - a(\psi^i, \psi^i) \right],
$$

(16)

$$
u - u^i = g^i_{m+1}(\psi^i - \psi^i), \quad J^i_2 = |g^i_{m+1}| \max_{x \in T^i} |\psi^i(x) - \psi^i(x)|
$$

(17)

hold true, where $g^i_{m+1}$ is determined in (14) and the function $y^i \in \hat{H}^i_n$ is defined by the condition

$$
a(y^i, v) = a(\psi^i, v) \quad \forall v \in \hat{H}^i_n.
$$

(18)

**Proof.** Since $H^i_n = \hat{H}^i_n$, we have from (4) and (15)

$$
a(z^i, v) = 0 \quad \forall v \in \hat{H}^i_n.
$$

(19)

for $z^i = u - u^i$. Let us prove the validity of relations (17). For $v = z^i - g^i_{m+1}(\psi^i - \psi^i) \in \hat{H}^i_n$, with due account of (18) and (19), we obtain $a(u, v) = a(z^i, v) - g^i_{m+1}a(\psi^i, \psi^i, v) = 0$. Since the quadratic form is positive on the nonzero function from $\hat{H}^i_n$, this relation holds if and only if $v = 0$. Hence, relations (17) hold true. Taking into account the symmetry of the form $a(u, v)$, we have

$$
2J^i_2 = a(u, v) - a(u^i, v^i) - 2h(z^i) = 2[a(u, z^i) - h(z^i)] - a(z^i, z^i) = 2[a(u, z^i) - h(z^i)] - a(z^i, v^i) + g^i_{m+1}a(z^i, \psi^i).
$$
where \( v = x^i + g^i_{m+1} \psi^i \in \tilde{H}_n \). Using (4) with the indicated function \( v \), relation (19), and the representation for \( z^i \), we obtain formula (16) for \( J^i_z \). The theorem is proved.

It remains to determine the quantities

\[
\hat{I}_1 = |s_{m+1}^i [b(\psi^i) - a(u, \psi^i)] - 0.5s_{m+1}^i a(\psi^i, \psi^i)|,
\hat{I}_2 = |s_{m+1}^i|,
\]

which are used for calculating the values of correction indicators.

### 3.3. Justifying the Choice of Correction Indicators

We define the bilinear form

\[
a^i(u, v) = \int_{T^i} \left( \sum_{\alpha, \beta = 1}^2 k_{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial x_\beta} + q u v \right) dx.
\]

The following lemma is valid by virtue of assumptions concerning the matrix \( k \) and the function \( q \).

**Lemma 1.** For the function \( v \in W^1_2(T^i) \) which is nonzero on \( T^i \) the relation \( a^i(v, v) = 0 \) holds if and only if \( q(x) = 0, v(x) = \text{const} \) for \( x \in T^i \).

We denote by \( \varphi_{ij}, 1 \leq j \leq 3 \), the basis functions associated with the vertices of the triangle \( T^i \). We determine the matrix \( C \) and the vector \( z \),

\[
C = \begin{pmatrix}
a^i(\varphi_{i1}, \varphi_{i1}) & a^i(\varphi_{i1}, \varphi_{i2}) & a^i(\varphi_{i1}, \varphi_{i3}) \\
a^i(\varphi_{i2}, \varphi_{i1}) & a^i(\varphi_{i2}, \varphi_{i2}) & a^i(\varphi_{i2}, \varphi_{i3}) \\
a^i(\varphi_{i3}, \varphi_{i1}) & a^i(\varphi_{i3}, \varphi_{i2}) & a^i(\varphi_{i3}, \varphi_{i3})
\end{pmatrix},
\]

and also define, in an ordinary way, the inner product of vectors, \( (\pi, v) = \pi^T v \).

**Lemma 2.** The matrix \( C \) is symmetric and nonnegative and the system of equations

\[
Cw = z
\]

is simultaneous. The matrix \( C \) is degenerate if and only if \( q(x) = 0 \) for \( x \in T^i \), and, in this case, the subspace \( \ker C \) is univariate and consists of vectors \( v = (c, c, c)^T \).

**Proof.** The symmetry of \( C \) is obvious and the nonnegativity follows from the relation

\[
a^i(v, v) = (Cv, v), \quad \text{where} \quad v(x) = \sum_{j=1}^3 v_j \varphi_{ij}(x), \quad v = (v_1, v_2, v_3)^T.
\]

Suppose that the matrix \( C \) is degenerate. Then there exists a nonzero vector \( v \) such that \( Cv = 0 \), and therefore the relation \( a^i(v, v) = 0 \) holds by virtue of (21). By Lemma 1, this is only possible when \( q(x) = 0 \) and \( v(x) = \text{const} \) for \( x \in T^i \). It follows that the subspace \( \ker C \) consists of vectors of the kind indicated in the lemma.

Suppose now that \( q = 0 \) on \( T^i \). By virtue of relation \( \varphi_{i1}(x) + \varphi_{i2}(x) + \varphi_{i3}(x) = 1 \), which is valid for \( x \in T^i \), we find that the relations

\[
Cv = (a^i(\varphi_{i1}, 1), a^i(\varphi_{i2}, 1), a^i(\varphi_{i3}, 1))^T = 0, \quad (z, v) = a^i(1, \varphi^i) = 0
\]

hold for the vector \( v = (1, 1, 1)^T \). It follows that \( C \) is degenerate and \( z \) belongs to the subspace \( \text{im} C \) which is orthogonal to \( \ker C \). Since \( z \in \text{im} C \), the system of equations (20) is simultaneous. The lemma is proved.

**Lemma 3.** Let the function \( y^i \in \tilde{H}_n \) satisfy condition (11). The relation

\[
a^i(y^i, y^i) = 0
\]

is valid.
holds if and only if
\[ a'(\varphi^i, v) = 0 \quad \forall v \in \tilde{H}_n. \]  
(23)

If condition (23) is fulfilled, then \( y'(x) = 0 \) for \( x \in \overline{G} \).

**Proof.** Suppose that (23) is fulfilled. Since the support of the function \( \varphi^i \) is concentrated on \( T^i \), it follows that \( a(\varphi^i, v) = a'(\varphi^i, v) = 0 \). Using the relation
\[ a(\varphi^i, y') = a(y', y') \]  
(24)
which follows from (11) for \( v = y' \), we obtain \( a(y', y') = 0 \). Since the form \( a(\cdot, \cdot) \) is coercive on the functions from \( \tilde{H}_n \), it follows that \( y'(x) = 0 \) for \( x \in \overline{G} \), and therefore relation (22) is valid.

Suppose that (22) is satisfied. Then either \( y' = 0 \) on \( T^i \) and, consequently, \( a(\varphi^i, y') = a'(\varphi^i, y') = 0 \), or by virtue of Lemma 1, \( q = 0 \), \( y' = \text{const} \) on \( T^i \) and therefore
\[ a(\varphi^i, y') = a'(\varphi^i, y') = \int_{T^i} q y' \varphi^i \, dx = 0. \]

By relation (24) we have \( a(y', y') = 0 \). Since the form \( a(\cdot, \cdot) \) is coercive on the functions from \( \tilde{H}_n \), it follows that \( y'(x) = 0 \) for \( x \in \overline{G} \). This relation and (11) imply (23). The lemma is proved.

**Lemma 4.** If the function \( y^i \in \tilde{H}_n \) satisfies condition (11), then the inequalities
\[ 0 \leq a(\varphi^i, y') \leq \gamma a(\varphi^i, \varphi') \]  
(25)
hold true. Here \( 0 \leq \gamma < 1 \) is a constant defined as follows: \( \gamma = 0 \) if (23) is satisfied, otherwise
\[ \gamma = \frac{[a'(\varphi^i, y')]^2}{a(\varphi^i, \varphi') a'(\varphi^i, y')} \]  
(26)

**Proof.** The left inequality is (25) follows from (24) and from the coerciveness of the form \( a(\cdot, \cdot) \). Let us prove the right inequality. If (23) is satisfied, then \( a(\varphi^i, y') = a'(\varphi^i, y') = 0 \), and the right inequality in (25) holds with \( \gamma = 0 \). Suppose now that condition (23) is not fulfilled. Then, by virtue of Lemma 3, we have \( a(y', y') \neq 0 \). Since \( 0 < a(y', y') \leq a(y', y') \) and \( a'(\varphi^i, \varphi') = a(\varphi^i, \varphi') \), the inequality
\[ [a'(\varphi^i, y')]^2 = \gamma a'(\varphi^i, \varphi') a'(y', y') \leq \gamma a(\varphi^i, \varphi') a(y', y') \]
with the value of \( \gamma \) indicated in (26) is valid. Hence, with due account of relation (24), we obtain the required inequality. The inequality \( \gamma \leq 1 \) follows from the Cauchy–Bunyakovsky inequality for integrals, the Cauchy inequality for sums, and the symmetry of the matrix \( k \). Since \( \varphi^i \notin \tilde{H}_n \), we have \( \gamma < 1 \). The lemma is proved.

In the following lemma we give an estimate for \( \gamma \) in terms of quantities which are independent of the function \( y^i \) and which can be found simultaneously with the calculation of the correction indicator.

**Lemma 5.** The estimate
\[ \gamma \leq \gamma^i = \frac{(z, w)}{a'(\varphi^i, \varphi^i)} < 1, \]
where \( w \) is the solution of system (20) if \( q \neq 0 \) on \( T^i \) and some one of its solutions if \( q = 0 \), holds true for \( \gamma \) from inequality (25).

**Proof.** If condition (23) is fulfilled, then \( z = 0 \). Therefore \( \gamma^i = 0 \) and coincides with the value of \( \gamma \) indicated in Lemma 4 for this case. Suppose now that (23) is not fulfilled. It follows from Lemma 3 that \( a'(y', y') > 0 \) in the case under consideration. From (20), (21) and the relation \( a'(\varphi^i, v) = (z, v) \), where
\[ v(x) = \sum_{j=1}^{3} v_j \varphi_{ij}(x), \ v = (v_1, v_2, v_3)^T, \]
we obtain
\[ \frac{[a'(\varphi^i, y')]^2}{a'(y', y')} \leq \max_{v: a'(v, v) > 0} \frac{[a'(\varphi^i, v)]^2}{a'(v, v)} = \max_{v: \varphi \in C} \frac{(Cw, v)^2}{(Cv, v)}. \]
Since the matrix $C$ is symmetric and nonnegative by virtue of Lemma 2, the inequality $(Cw, v)^2 \leq (Cw, w)(Cv, v)$ holds true. From this inequality, from the estimate obtained above and from (26) follows a relation for $\gamma^t$. Note that in the case of a degenerate matrix $C$ we can take some solution of system (20) as $w$ since any two of its solutions differ by an arbitrary vector belonging to $\ker C$ and the vector $z$ is orthogonal to this subspace.

Let us show that $\gamma^t < 1$. Using $w = (w_1, w_2, w_3)^T$, we construct a function $w(x) = \sum_{j=1}^3 w_j \varphi_{ij}(x) \neq 0$.

Since $(x, w) = a^t(\varphi^t, w) = a^t(w, w)$, there exists a representation for $\gamma^t$ obtained from (26) upon the replacement of $y^t$ by $w$. The proof of the lemma is obviously complete.

The following theorem is implied by Lemmas 4 and 5.

**Theorem 3.** If condition (23) is fulfilled, then the correction indicators $I_1$ and $I_2$, defined by relations (13), are exact, i.e., $I_1 = I_2^t$, $\alpha = 1, 2$. If (23) is not fulfilled, then the inequalities

$$I_1 \leq J_1^t \leq \frac{1}{1 - \gamma^t} I_1^t, \quad I_2 \leq |w_{k+1}| \leq \frac{1}{1 - \gamma^t} I_2^t,$$

where the constant $\gamma^t$ is defined in Lemma 5, are valid.

**Remark 1.** On the triangle $T^t$ only three basis functions $\varphi_{ij}$ have nonzero values. Therefore condition (23) is equivalent to the requirement that $a^t(\varphi^t, \varphi_{ij}) = 0$ for $\varphi_{ij} \in \bar{H}_n$. Note that the basis function $\varphi_{ij}$ associated with the vertex $x_i$, belongs to $H_\omega$, if $x_i \in \omega$. Consequently, if all vertices of the triangle lie on $\partial C$, then condition (23) is fulfilled for this triangle. For all other triangles at least one of the functions $\varphi_i$ belongs to $\bar{H}_n$. We can easily reformulate the given above in terms of coordinates of the vertices of the triangle $T^t$ and coefficients $k_{a, \beta}$ and $q$ using the definitions of the basis functions $\varphi_{ij}$ and $\varphi^t$.

**Remark 2.** It is easy to show that if the coefficients $k_{a, \beta}$ are constant on $T^t$ and $q = 0$, then the correction indicators $I_1$ and $I_2$ are exact.

4. THE PROCEDURE OF MESH REFINEMENT

The procedure of refinement consists of two stages, namely, a stage of addition of new nodes and a stage of constructing the graph of a new triangulation. The first stage consists of two steps. At the first step the set of nodes is complemented by new nodes which are placed on some sides of triangles from the list $List$, and on the line segments of the boundary from the list $List_0$. At the second step, for ensuring the conformity of the new grid, the obtained set can be expanded by the addition of new nodes which are also placed on the sides of triangles of the current triangulation.

When constructing the graph of the new triangulation, we shall proceed from the requirement of preservation of the grid conformity. The second important requirement is that in the process of mesh refinement no triangles with very large or small angles should appear. It is well known [22, 23] that the appearance of a triangle with a large angle leads to a decrease of accuracy of the finite-element approximation and the appearance of a triangle with a small angle leads to a growth of the condition number of the matrix of the system of equations (5). The exactness of the finite-element solution also depends on the smoothness of the grid (adjacent triangles must not differ essentially from one another by their area).

Below we describe two procedures of mesh refinement $Ref_1$ and $Ref_2$ which differ by the realization of the first step of the stage of addition of new nodes. The first of them is close to the procedure described in [24] and the second one is a combination of the bisection method [25] and the procedure indicated above. In the description of the procedures we use the notion of the larger side of the triangle, i.e., the side of a larger length. If two or three sides of a triangle have a maximal length, then one of them is chosen and fixed as the larger side.

I. THE STAGE OF ADDITION OF NEW NODES

1. **The first step.** Procurement of the elements from the lists $List_0$ and $List_1$.

   (a) The new nodes are placed on the sides of all segments from the list $List_0$. The numbers of triangles corresponding to these segments are put into the auxiliary list $List$. 

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(b) For each triangle from the list \( \text{List}_t \) the new nodes are placed either at the middle of all sides of the triangle (procedure \( \text{Ref}_t \)) or at the middle of the larger side (procedure \( \text{Ref}_l \)). In the first case, the list \( \text{List}_t \) is complemented by the numbers of triangles adjacent to the triangles from \( \text{List}_t \) and in the second case, the list \( \text{List}_t \) is complemented by all triangles from \( \text{List}_t \).

2. **The second step.** For each triangle from the list \( \text{List}_t \) the following recursive procedure is realized.

(a) If there is no new node at the middle of the larger side of the triangle being processed, then it should be placed there.

(b) If there is a triangle adjacent, along the larger side, to the triangle being processed (this side of the triangle is not a segment of the boundary), then, in the case where there is no new node on the larger side of the adjacent triangle under consideration, it should be placed there and item 2b should be performed for processing this adjacent triangle, otherwise we should pass to item 2a.

Note that if new nodes appear on the sides of the triangle after this stage, then one of them necessarily lies on the larger side of the triangle.

II. The stage of constructing the graph of a triangulation
1. The edges of the graph of the current triangulation which do not contain new nodes remain in the graph of the new triangulation. Each edge of the graph of the current triangulation which contains a new node is replaced by two edges which connect this node with the nodes which are endpoints of the edge being replaced.

2. In order to ensure the conformity of the new grid, it remains to complement the graph of the triangulation by new edges having processed all triangles as follows. If there is only one new node on the boundary of the triangle, then it is connected with the opposite vertex of the triangle. If there are exactly two new nodes on the boundary of the triangle, then the node lying on the larger side is connected to the opposite vertex of the triangle and to the second new node. If there are three new nodes on the boundary of the triangle, then they are connected to one another.

The rule of triangulation rearrangement given above makes it possible to avoid the appearance of triangles with very small or very large angles.

5. ITERATION METHOD

Let \( \mathcal{W}^{(k)} \) be a grid at the \( k \)th iteration and suppose that \( u^{(k)} \) is the corresponding approximate solution of problem (1), (2) on this grid. The iteration method is described as follows.

1. **Initialization.** An initial triangulation of the domain \( G \) is defined and an empty list \( \text{List}_t \) is formed.

2. **Construction of the initial grid.** The construction of the grid \( \mathcal{W}^{(0)} \) is carried out iteratively.

(a) For each new line segment \( \mathcal{IG}_i \) (first all segments are considered to be new) the value of \( R^* \) is calculated from relation (7) and the list \( \text{List}_t \) is formed in accordance with the rule given in item 3.

(b) If the list turns out to be empty, then we pass to item 3, otherwise we construct a new grid using the procedure of mesh refinement described in item 4 and pass to item 2a.

3. **Initial step.** A system of equations (5) is constructed, \( u^{(0)} \) is found, and the value of functional \( J(u^{(0)}) \) is calculated with the use of relation (6).

4. **Iteration step.** The following stages should be performed for \( k = 1, 2, \ldots \).

(a) For each new triangle \( T^* \) (first all triangles are considered to be new) the value of the correction indicator \( I^* \) of the chosen type is found and the list \( \text{List}_t \) is formed according to the rule of choice of triangles described in item 3.

(b) Using the chosen procedure of adaptive mesh refinement (see item 4), a new grid \( \mathcal{W}^{(k)} \) is constructed. Then a system of equations (5) is constructed, an approximate solution \( u^{(k)} \) is found, and \( J(u^{(k)}) \) is calculated with the use of relation (6).

(c) **Test for convergence.** If the conditions

\[
J(u^{(k-1)}) - J(u^{(k)}) \leq atol + rtol |J(u^{(k)})|,
\]

where \( atol \) and \( rtol \) are given tolerances, are fulfilled, then the iterative process is completed, otherwise \( k \) is increased by 1 and we pass to item 4a.
Remark. In the test for convergence we can also use other criteria of completion of the iterative process, in particular, criteria based on the use of a posteriori error estimates $z^{(k)} = u^{(k)} - u^*$. Different methods of obtaining these estimates are considered in [12–15].

6. RESULTS OF NUMERICAL EXPERIMENTS

The aim of the experiments was the estimation of the efficiency of the method which uses the strategies and procedures of adaptive mesh refinement proposed in the work as compared to the method which uses the strategy of uniform mesh refinement.

Problem (1), (2), where $\Omega = \{0 < x_1 < 1, 0 < x_2 < 1\}$, was used as the test problem,

$$k_{11}(x) = k_{22}(x) = \frac{\varepsilon}{4} \sum_{\alpha = 1}^{2} [e^{-\alpha x_1} + e^{\varepsilon(\alpha - 1)}], \quad \varepsilon = 10^{-4},$$

$$k_{12}(x) = k_{21}(x) = 0, \quad q(x) = e^{x_1(1-x_2)+x_2(1-x_1)},$$

$f(x)$ and $u_0(x)$ were chosen such that the function

$$u^*(x) = 3 + \sum_{\alpha = 1}^{2} \left[ e^{-(2\alpha - 1)x_1}/\sqrt{\varepsilon} - e^{-(2\alpha - 1)x_2}/\sqrt{\varepsilon} - e^{(\alpha - 1)/\sqrt{\varepsilon}} \right]$$

was the classical solution of the differential problem (3) which had a strong exponential growth in the off-boundary domain and in the neighborhood of the point $x = (\frac{1}{2}, \frac{1}{2})$. For the test problem with an accuracy of $10^{-10}$ we have $J(u^*) = -7.1573091002796$.

For the approximate calculation of integrals necessary for the generation of the system of equations (5) we used the version of the Romberg procedure constructed for triangles. At each iteration, only those elements of the matrix $A$ and the right-hand side $b$ were calculated which were connected with the added basis functions. The solution of system (5) at each iteration was found approximately, by the explicit conjugate gradient method. When constructing the initial approximation for this internal iteration method, we used the solution found on the preceding grid and interpolated to the new grid.

In the criterion of choice of the line segments of the boundary $\partial \Omega$ we chose $\mu = 1/16$ (by virtue of the quadratic dependence of the interpolation error on the step in the process of piecewise-linear interpolation of the smooth function on a uniform grid) and set $\varepsilon^2 = 10^{-5}$. The value $\theta = 0.2$ was used in the criterion of choice of triangles $T^k$.

The results characterizing the convergence of the approximate solution $u^{(k)}$ to the exact solution $u^*$ for the method with adaptive $S^k$ strategy and the Ref$S^k$ procedure of mesh refinement as well as for the method with the strategy of uniform mesh refinement are given in Tables 1 and 2.

Figure 1 shows the starting triangulation of the domain and the initial grid constructed for adaptive iteration method. Figure 2 shows a grid constructed at the last iteration and the approximate solution of problem (1), (2) obtained on this grid.

In these tables (the superscript $k$ is omitted) we used the following notation:

$$r^k_\omega = \max_{x \in \Omega^{(k)}} \left| \frac{u^{(k)}(x) - u^*(x)}{u^*(x)} \right|, \quad r^k_\Omega = \max_{x \in \Omega^{(k)}} \left| \frac{u^{(k)}(x) - u^*(x)}{u^*(x)} \right|, \quad r^k_j = \frac{J(u^{(k)}) - J(u^*)}{|J(u^*)|},$$

where $\Omega^{(k)} = \bigcup_{i=1}^{n^{(k)}} \Omega_i^{(k)}$ and $\Omega_i^{(k)}$ is the set of nodes of the auxiliary grid in the triangle $T^k$. The use of the set $\Omega^{(k)}$ allows us to estimate the error at a large number of points lying between the nodes of the grid $\omega^{(k)}$ and simulate, in this way, the error estimate in the norm of the space $C(\Omega)$. The problem was considered to be solved when the relative error of the obtained solution $r^k_\Omega$ did not exceed $10^{-9}$.

For the standard finite element method which uses the strategy of uniform mesh refinement the initial grid was constructed as follows. In the domain $\Omega$ we introduced a square grid with four nodes along each
Table 1
Method used on adaptive mesh refinement

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Table 2
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The mesh of the grid was divided by the diagonal into two triangles. In the mesh refinement procedure we divided each triangle into four similar triangles by connecting the new nodes lying at the middle of the sides of the triangles.

Experiments have shown that the most efficient of the considered test problem was the method which used the adaptive strategy $S_2$ and the procedure Ref2 of mesh refinement. The standard finite element method with uniform mesh refinement is considerably inferior relative to any one of the versions (combination of strategy and mesh refinement procedure) of the method proposed in this work both as concerns the required memory and the time taken by problem solution. It requires 25 times more nodes and 10 times more time than the best adaptive method proposed in this work.
Fig. 1
Triangulation of the domain (a) and the initial grid (b).

Fig. 2
A grid constructed at the last iteration (a) and the solution obtained at it (b).

REFERENCES


30 March 2005